The Equational Theories Project: Advancing Collaborative Mathematical Research at Scale

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(Tao's) Motivation

- Traditional research: small teams (1–5 experts), hard to scale
- Larger-scale collaborations face verification challenges
- Public contributions often infeasible due to complexity
- AI tools: useful, but can hallucinate → need verification

Proof Assistants as Enablers

- Proof assistants (e.g. Lean) allow modular verification
- Contributions can be verified automatically
- Enables participation from:
  - Professional mathematicians
  - General public
  - Automated tools (ATPs, AI)
- Prior successes: formalization of known results (e.g. PFR)

Towards New Mathematics

- Polymath projects: precedent for online collaboration
- Bottleneck: human moderators verifying contributions
- Adding proof assistants removes this barrier
- Goal: not just formalizing existing results, but exploring *new* mathematics collaboratively

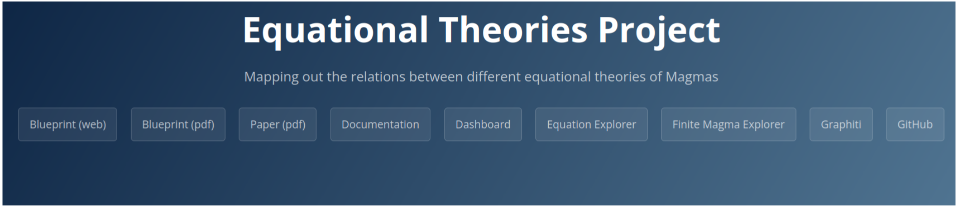
Exploring Classes of Problems

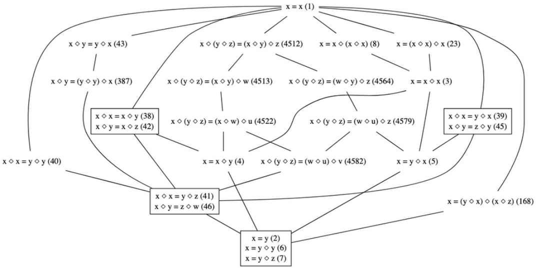
- Aim: explore *families* of mathematical problems at once
- Modular, repetitive tasks $\rightarrow$ well-suited for:
  - Crowdsourcing
  - Automated tools
- Benefits: large datasets for benchmarking, faster intuition
- Analogy: Busy Beaver Challenge, GIMPS

The Equational Theories Project (ETP)

- Pilot project testing this paradigm
- Inspired by MathOverflow question + Mastodon discussion
- Launched Sept 2024
- Goal: Determine complete implication graph of magma laws

The Equational Theories Project (ETP)



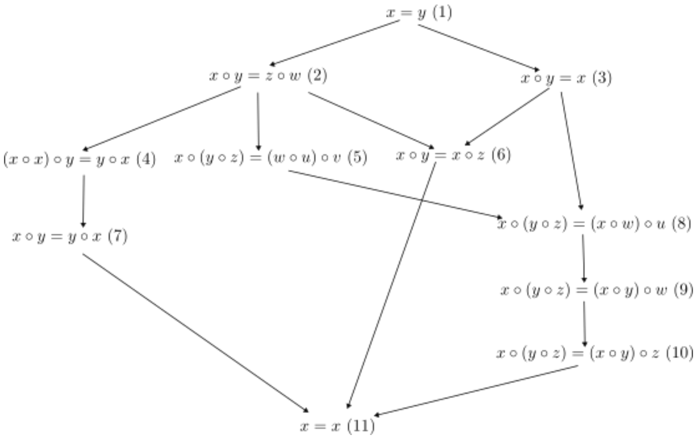


Collaboration Model

- Infrastructure:
  - Lean proof assistant
  - GitHub repository
  - Zulip chat for discussion
- Contributions:
  - Human participants (expert + public)
  - Automated tools (ATPs, AI)
- Key: modular, verifiable pieces

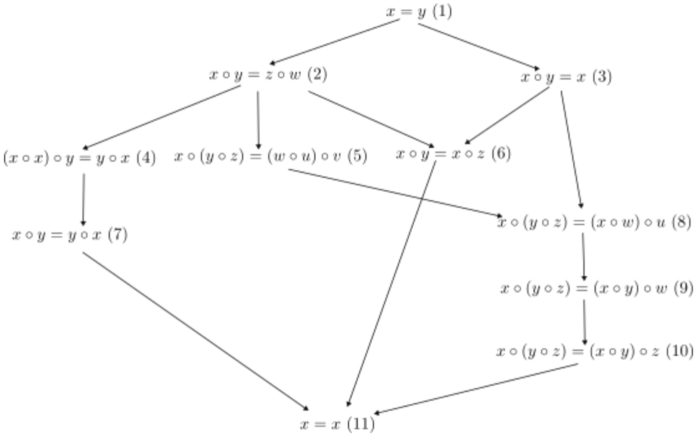
Background: Magmas and Laws

- A **magma**: a set with a binary operation $\circ$
- Equational axioms: equalities built from $\circ$ and variables
- Examples:
  - Commutativity: $x \circ y = y \circ x$
  - Associativity: $(x \circ y) \circ z = x \circ (y \circ z)$
  - Singleton: $x = y$
- Identity axiom $e \circ x = x$ excluded (involves constant e)



The Goal

- Up to trivial equivalences, there are 4694 different laws with ≤ 4 applications of $\circ$
- Define the implication graph by drawing an edge $E_n \models E_m$ if every magma obeying E_n also obeys E_n
- Goal: Determine the entire Hasse diagram
- There are $4694(4694-1) = 22\,028\,942$ implications to prove or disprove



Example entailment proof

Proposition 1 Equation4 implies Equation7.

Proof: Suppose that G obeys Equation4, thus

$$(x \circ x) \circ y = y \circ x \qquad (1)$$

for all $x, y \in G$. Specializing to $y = x \circ x$, we conclude

$$(x \circ x) \circ (x \circ x) = (x \circ x) \circ x$$

and hence by another application of (1) we see that $x \circ x$ is idempotent:

$$(x \circ x) \circ (x \circ x) = x \circ x. \qquad (2)$$

Now, replacing x by $x \circ x$ in (1) and then using (2), we see that

$$(x \circ x) \circ y = y \circ (x \circ x),$$

so in particular $x \circ x$ commutes with $y \circ y$:

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ (x \circ x). \qquad (3)$$

Also, from two applications (1) one has

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ x = x \circ y.$$

Thus (3) simplifies to $x \circ y = y \circ x$, which is Equation7. $\square$

The Goal

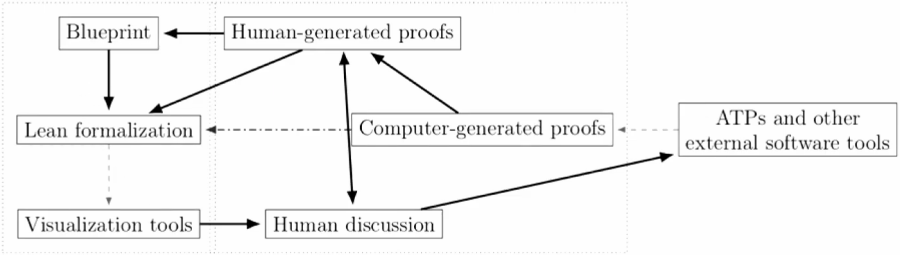
- Many of the implications or anti-implications are easy to work out by hand. But this does not scale.
- Knowing some parts of the graph can help determine others, for example by transitivity.
- The graph also has a *duality symmetry* under the involution from $(x, y) \mapsto x \circ y$ to $(x, y) \mapsto y \circ x$.

The Goal

- In general, the problem is known to be *undecidable*
- ATPs can decide many specific implications, but are not guaranteed to succeed in finite time
- A few specific laws are well studied: e.g. $x = y(z(x(yz)))$ characterizes abelian groups under subtraction. But the vast majority of laws had no established literature.
- Collaboration needed to resolve all 22 028 942 implications
- Formal verification needed to *verify* all 22 028 942 implications

Formalization Workflow

- Partially run like a regular Lean project + blueprint
- Automation + human contributions integrated via GitHub
- Zulip chat used for coordination and discussion



Methods

- Positive Implications ($E \models E'$):
  - Hand proofs
  - Rewriting rules
  - Duality & preorder structure
  - ATPs like `Vampire`, `Z3`
- Negative Implications ($E \not\models E'$):
  - Syntactic invariants
  - Small finite magmas
  - Linear / Translation-Invariant Models
  - Greedy construction methods
  - Twisting Semigroup Models
  - Ad-hoc

Zulip highlights

Equational > 1648 !=> 206

Bernhard Reinke 2:46 PM

Hello, I think I have a counterexample that shows that 1648 does not imply 206. The magma is supported on the integers and defined as follows

$$x \diamond y = x - sign(y - x)$$

In other words, $x \diamond x = x$, and $x \diamond y = x + 1$ if $x > y$, and $x \diamond y = x - 1$ if $x < y$.

Now this satisfies 1648: it is clear if $x = y$, otherwise consider the case $x > y$, then

$$(x \diamond y) \diamond ((x \diamond y) \diamond y) = (x+1) \diamond ((x+1) \diamond y) = (x+1) \diamond (x+2) = x$$

the case $x < y$ is analogous. But this magma does not satisfy 206:

indeed, for $x = 0$, $y = -1$ we have

$$(x \diamond (x \diamond y)) \diamond y = (0 \diamond 1) \diamond -1 = -1 \diamond -1 = -1 \neq 0 = x$$

🎉 4

Zulip highlights

Equational > **1648 !=> 206** OCT 14, 2024

Shreyas Srinivas EDITED 3:40 PM

Here's a long flow-of-thought proof. I think this can be golfed down significantly if `split` allowed naming the hypothesis and I used more mathlib defs:

```
import Mathlib.Tactic

class Magma (α : Type _) where
  op : α → α → α

infix:65 " ◇ " => Magma.op

abbrev Equation1648 (G: Type _) [Magma G] := ∀ x y : G, x = (x ◇ y) ◇ ((x ◇ y) ◇ y)

abbrev Equation206 (G: Type _) [Magma G] := ∀ x y : G, x = (x ◇ (x ◇ y)) ◇ y


def sign (x : ℤ) :=
  if x = 0 then 0 else if x < 0 then -1 else 1

theorem Equation1648_not_implies_Equation206 : ∃ (G: Type) (_: Magma G), Equation1648 G ∧ ¬
  let instMagmaInt : Magma ℤ := {
    op := fun x y => x - sign (y -x)
  }
  use ℤ, instMagmaInt
  simp [Equation1648, Equation206]
  constructor
  · intro x y
```

Zulip highlights

Equational > Refutations using Z3 SEP 27, 2024

Daniel Weber EDITED 7:33 PM

Using Z3 I was able to refute `Equation4283[x ∘ (x ∘ y) = x ∘ (y ∘ x)] => Equation4358[x ∘ (y ∘ z) = x ∘ (z ∘ y)]`:

x	0	1	2	3	4	5	6
0	1	1	1	4	1	6	1
1	1	1	3	1	1	1	1
2	1	5	1	1	5	1	5
3	1	1	1	1	1	1	1
4	1	1	3	1	1	1	1
5	1	1	1	1	1	1	1
6	1	1	3	1	1	1	1

With `x = 0, y = 1, z = 2`

Zulip highlights

Equational > **1076 !=> 3** — OCT 14, 2024

Daniel Weber said: — 7:08 AM

> Only the rules are important, right? Once we have them I think it should be immediate how R' should be defined, if we restrict it to only differ from R in $R'(a, b, c)$ and $R'(c, \_, \_), R'(\_, c, \_)$ (and any values we have to set to satisfy old rules should be possible to simply convert to new rules)

This is false - in 1692 there's the rule $a \diamond a = x, b \diamond a = a, a \diamond b = c \to a \diamond x = c$, I need to think for a bit how to account for that

EDITED — 8:17 AM
It succeeded on 118, 124, 476, 503, 677, 707, 713*, 883, 906, 1112, 1113, 1289*, 1447. The * are equations it seemingly succeeded on, but couldn't prove that all rules are preserved. I'll try to run it on the remaining equations with more time

Terence Tao — 4:11 PM
Thats a pretty good success rate!

4:54 PM
When you are done with your run, can you update `Conjectures.lean` with all the implications that this procedure will likely resolve (presumably in the negative), so that it is

Zulip highlights

Equational > FINITE: The Lean+Duper implications DEC 5, 2024

Amir Livne Bar-on EDITED 9:31 PM

Looking at the Duper proofs of 1167=>X in https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Austin.20pairs/near/481257624, many of them go through 4689 (spelled "eq2693" in DuperConjectures1.lean). This was transitively reduced to the conjecture 1167=>4615.

Here's a proof for that one:

Equation 1167 is $L_y L_{z \diamond Sy} = I$, while equation 4615 is $L_{Sx} = L_{z \diamond x}$. For x s that are squares $x = Sa$ we have $L_{z \diamond x} = L_a^{-1}$, which is independent of z so 4615 holds.

But every element in 1167 is a square: $Sx = L_x x$, so $x = L_{z \diamond Sx} Sx$ for any z. We can take for example $a = L_{Sx \diamond Sx} Sx$ for the above.

👍 Terence Tao, Michael Bucko

Blueprint proofs

For particularly hard or intricate proofs, the blueprint was used to write a Latex version of the proof first

Now we seek to enlarge a partial solution. We first make an easy observation:

Proposition 18.13. (Enlarging L_0')✓

> *Suppose one has a partial solution in which $L_0'x$ is undefined for some $x \in N$. Then one can extend the partial solution so that $L_0'x$ is now defined.*

Proof ▾

By axiom (i''), $L_0'(R_0')^n x$ is undefined for every integer n. Let $d = E_m$ be a generator of SM that does not appear as a component of any index of any of the generators e_a appearing anywhere in the partial solution; such a d exists due to the finiteness hypotheses. We set $L_0'x := e_d$, and then extend by Equation 6, Equation 7, thus

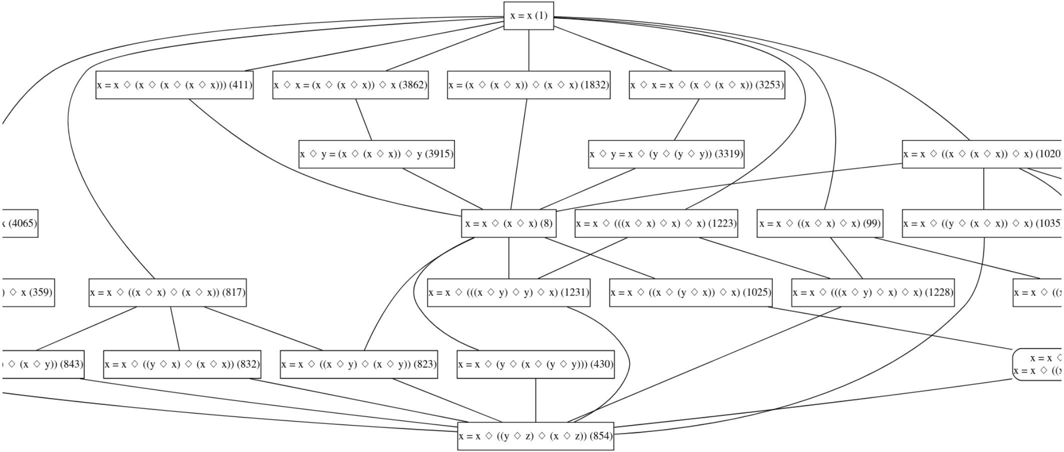
$$L_0'(R_0')^n x := (R_0')^n e_d$$

and

$$L_0'(R_0')^n e_d := (R_0')^{n-1} x.$$

Because of the new nature of d, no collisions in the partial function L_0 are created by

Visualization tools: Graphiti



Visualization tools: Equation Explorer

Equation Details

Back to List

Equation854[x = x ◇ ((y ◇ z) ◇ (x ◇ z))]

(Dual equation: Equation2712[x = ((y ◇ x) ◇ (y ◇ z)) ◇ x])
(Visualize implies and implied by of the equation, or see 1, 2, 3 graph edges away)
(Size of smallest non-trivial magma: 2 (Explore))

☑ Hide equivalent equations ☑ Treat conjectures as unknown ☐ Display the finite graph ☐ Show only explicit proofs

This equation implies (=>):

Implies	Does not imply	Unknown
Equation1[x = x] Try This! Show Proof	**Equation2[x = y] (+ 1495 equiv.)** Try This! Show Proof	None
Equation8[x = x ◇ (x ◇ x)] Try This! Show Proof	**Equation3[x = x ◇ x]** Try This! Show Proof	
Equation99[x = x ◇ ((x ◇ x) ◇ x)] Try This! Show Proof	**Equation4[x = x ◇ y] (+ 70 equiv.)** Try This! Show Proof	
Equation101[x = x ◇ ((x ◇ y) ◇ x)] (+ 1 equiv.) Try This! Show Proof	**Equation5[x = y ◇ x] (+ 70 equiv.)** Try This! Show Proof	
Equation359[x ◇ x = (x ◇ x) ◇ x] Try This! Show Proof	**Equation9[x = x ◇ (x ◇ y)] (+ 8 equiv.)** Try This! Show Proof	
Equation378[x ◇ y = (x ◇ y) ◇ y] Try This! Show Proof	**Equation10[x = x ◇ (y ◇ x)] (+ 5 equiv.)** Try This! Show Proof	

This equation is implied by (<=):

Implied by	Not implied by	Unknown by
Equation2[x = y] (+ 1495 equiv.) Try This! Show Proof	**Equation1[x = x]** Try This! Show Proof	None
Equation4[x = x ◇ y] (+ 70 equiv.) Try This! Show Proof	**Equation3[x = x ◇ x]** Try This! Show Proof	
Equation433[x = x ◇ (y ◇ (x ◇ (z ◇ y)))] Try This! Show Proof	**Equation5[x = y ◇ x] (+ 70 equiv.)** Try This! Show Proof	

Visualization tools: Finite Magma Explorer

Operation table:

```
0 1 2
1 2 0
0 1 1
```

Process

Satisfies:

- 1 - x = x
- 151 - x = (x ◇ x) ◇ (x ◇ x)
- 2441 - x = (x ◇ ((x ◇ x) ◇ x)) ◇ x

Refutes:

- 2 - x = y with {"x":0,"y":1}
- 3 - x = x ◇ x with {"x":1}
- 4 - x = x ◇ y with {"x":0,"y":1}
- 5 - x = y ◇ x with {"x":0,"y":1}
- 6 - x = y ◇ y with {"x":0,"y":1}
- 7 - x = y ◇ z with {"x":0,"y":0,"z":1}

Filter: number or expression

All implications refuted by this magma are already known.

Dashboard

The implication graph is **100.00000%** complete.

An implication is considered *explicitly true* or *explicitly false* if we have a proof of the corresponding proposition formalised in Lean. It is *implicitly true* or *implicitly false* if the proposition can be derived by taking the reflexive transitive closure of explicitly proven implications.

Our current counts of implications in each of those categories are:

explicitly true	implicitly true	explicitly false	implicitly false	no proof
10,657	8,167,622	586,925	13,268,432	0

The *no proof* column above represents work that we still need to do. Among the *no proof* implications, we have the following conjecture counts:

explicitly true	implicitly true	explicitly false	implicitly false	no conjecture
0	0	0	0	0

The implication graph is **100.00000%** complete if we include conjectures.

Dashboard

Finite graph

Some implications are true specifically only for finite magmas.

The finite implication graph is **99.99999%** complete.

explicitly true	implicitly true	explicitly false	implicitly false	no proof
10,750	8,168,349	586,220	13,268,315	2

The finite implication graph is **99.99999%** complete if we include conjectures.

explicitly true	implicitly true	explicitly false	implicitly false	no conjecture
0	0	0	0	2

Finite implication graph

- Recall: $E_n \models E_m$ if every magma obeying E_n also obeys E_n
- Let $E_n \models_{fin} E_m$ if every *finite* magma obeying E_n also obeys E_n
- This is not the same, as there are some cases where every $E_m \models E_n$ counterexample is infinite

The remaining open question

Theorem (Open problem)

Does the law E_{677}*:* $x = y \circ (x \circ ((y \circ x) \circ y))$ *imply the law* E_{255}*:* $x = ((x \circ x) \circ x) \circ x$ *for finite magmas?*

LLMs need not apply

- The ETP made extensive use of “good old-fashioned AI” in the form of ATPs
- But LLMs made only modest contributions:
  - Building visualization tools
  - code completion
  - guessing a rewriting system for a specific law from similar examples
- The implication graph appears to have some structure, such that a neural network may be able to predict it with high accuracy given a portion of the graph. This is still speculative however, and we did not use ML for this during the project.

The End

https://teorth.github.io/equational_theories