Bol-Moufang rings and things, yet again

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AITP 2025, Aussois

Quasigroups

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A *quasigroup* is a set with three binary operations, $\cdot, /, \backslash$, satisfying:

$$x \cdot (x \backslash y) = y = (y / x) \cdot x$$
$$x \backslash (x \cdot y) = y = (y \cdot x) / x$$

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2. *Moufang loops*: $x \cdot (y \cdot (x \cdot z)) = ((x \cdot y) \cdot x) \cdot z$

2. *(left) Bol loops*: $x \cdot (y \cdot (x \cdot z)) = (x \cdot (y \cdot x)) \cdot z$

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Bol-Moufang

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Bol-Moufang identity

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Bol-Moufang identity

60 such identities

Ruth Moufang (1905–1977)

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Gerrit Bol (1906–1989)

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Toward a hard problem

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Find/classify the varieties of loops axiomatized by a single identity of Bol-Moufang type.

The varieties of loops of Bol-Moufang type (Fenyves, '64; P. and Vojt. '05)

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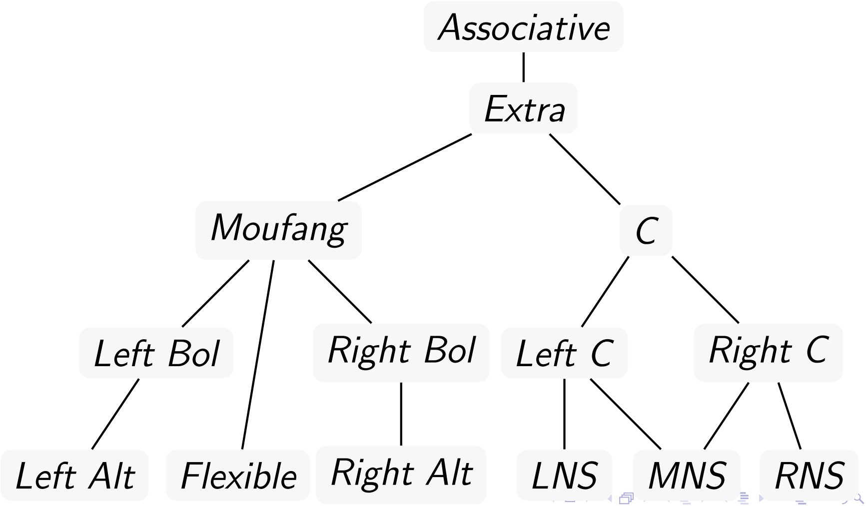

Analogies and new territory

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Well known and investigated varieties (of these 14):

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1. *Groups,*

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1. Groups, 2. extra loops,

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1. Groups, 2. extra loops, 3. Moufang loops, 4. left Bol loops, 5. right Bol loops, 6. flexible loops,

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Well known and investigated varieties (of these 14):

1. Groups, 2. extra loops, 3. Moufang loops, 4. left Bol loops, 5. right Bol loops, 6. flexible loops, 7. left alternative loops,

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Analogies between left and right sides of the Hasse diagram; i.e., between Moufang loops and C loops, and between left Bol loops and left C loops, and between right Bol loops and right C loops.

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Unexplored (and alas, very thinly structured) territory: *10. left C loops, 11. right C loops, 12. left nuclear square loops,*

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$$x(y \cdot yz) = (x \cdot yy)z$$

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Theorem

Let L be a left C loop. Then, $N_\lambda(L) = N_\mu(L)$ is normal in L.

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Theorem (P. and Vojt. '06)

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Theorem (Kinyon and P. '24)

The quotient of a left C loop by its left nucleus is a left Steiner loop.

Nuclear squares

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Wow, there's almost no structure! Where to even start?!

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Theorem (Kinyon and P., '24)

Let L be a loop with left and middle nuclear squares. Then $N_{\lambda,\mu}(L)$ is normal in L.

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Let L be a loop with left and middle nuclear squares. Then $N_{\lambda,\mu}(L)$ is normal in L.

Corollary (Kinyon and P., '24)

Let L be a loop with nuclear squares. Then $\mathrm{Nuc}(L)$ is normal in L.

Bruck loops and the AIP

Automorphic inverse prop. (AIP): $(xy)^{-1} = x^{-1}y^{-1}$.

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Recall: analogy between left Bol loops and left C loops.

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Note: right C loops are mirror images of (i.e., dual to) left C loops. That is: every theorem about left C loops dualizes to a theorem about right C loops.

AIP left C loops

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Lemma (Kinyon and P., '24)

Let L be a loop with squares in two nuclei. If L has the AIP or ES, then it has central squares (and hence, both AIP and ES).

AIP left C loops

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Lemma (Kinyon and P., '24)

Let L be a loop with squares in two nuclei. If L has the AIP or ES, then it has central squares (and hence, both AIP and ES).

Theorem (Kinyon and P., '24)

Let L be a finite loop with squares in two nuclei and the AIP/ES. Then

$$L = E \times O$$

where E is a loop in which each element has order a power of two and O is a loop of odd order.

"Rings"

“Rings”

$(R, +, \cdot)$ is a “*ring*” if

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2. $\cdot$ distributes over $+$ (left and right)
3. $(R, \cdot)$ is a semigroup

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Unital ring: ring with two-sided identity element.

The varieties of unital rings of Bol-Moufang type (P. and Rowe, '24)

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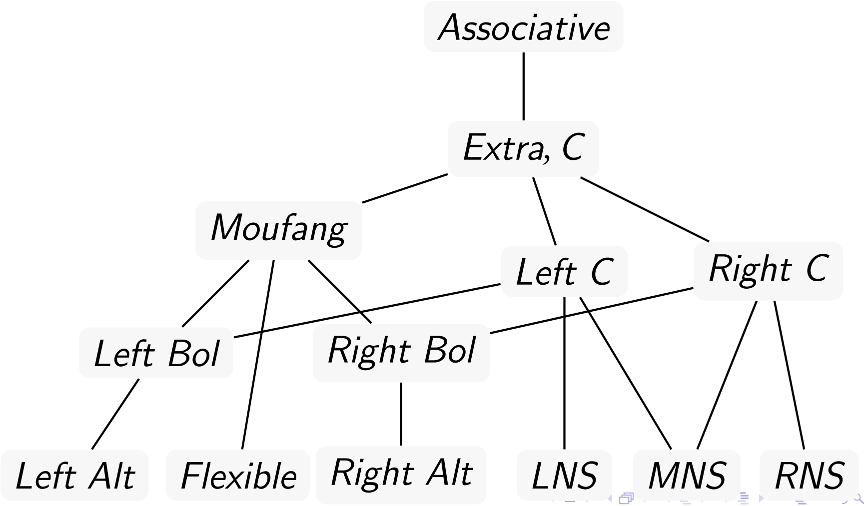
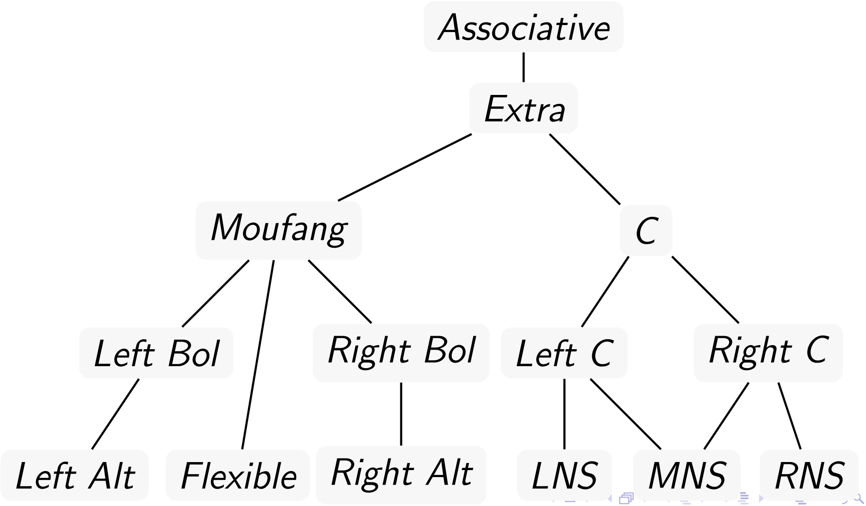


The varieties of loops of Bol-Moufang type (Fenyves, '64; P. and Vojt. '05)



A few comments

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Proofs (implications)

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Distinguishing examples

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Distinguishing examples, magma rings

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Characteristic

A few comments

Proofs (implications)

Distinguishing examples, magma rings

Characteristic 2

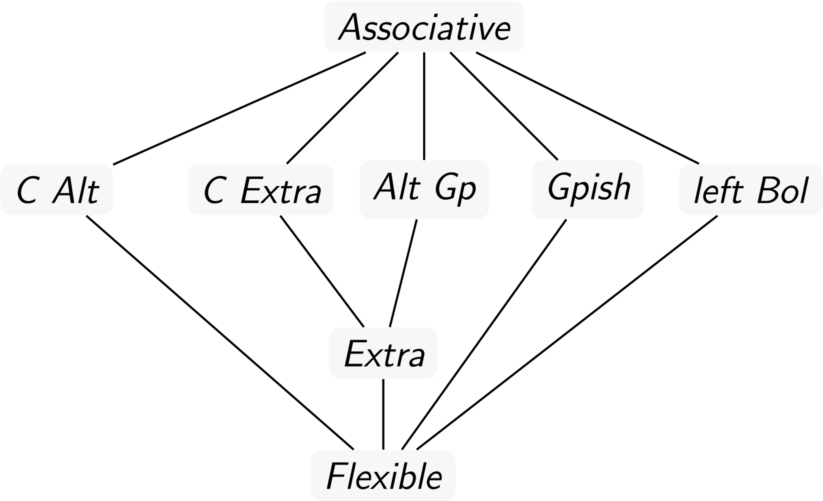
Lie Rings

Lie Rings

A *Lie ring* is a ring with a bracket operation, $[\ ,\ ]$, satisfying:

1. $[x, x] = 0$,
2. $[x, y] + [y, x] = 0$,
3. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

The varieties of lie rings of Bol-Moufang type


Associative
C Alt
C Extra
Alt Gp
Gpish
left Bol
Extra
Flexible

Thanks!

Thank you for your kind attention.