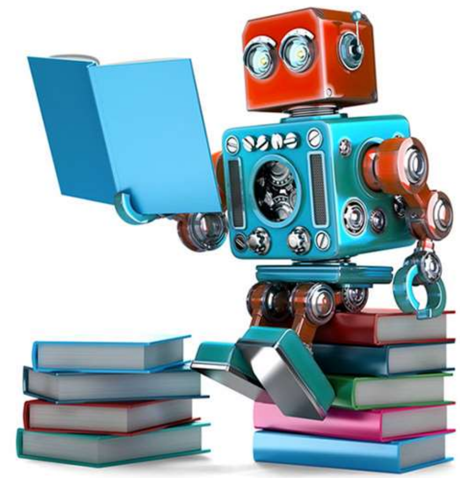
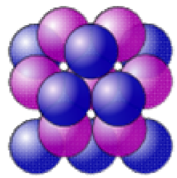
Computer-assisted mathematics



- Kepler conjecture (sphere packing)

  J.Kepler (1611) ...  ... T.Hales (1998)

- Robbins conjecture (abstract algebra) W.McCune (1996)

- 4-color Conjecture → Theorem K.Appel, W.Haken (1977)

  F.Guthrie (1852), A.Cayley (1878), A.Kempe (1879), P.Tait (1880), ...

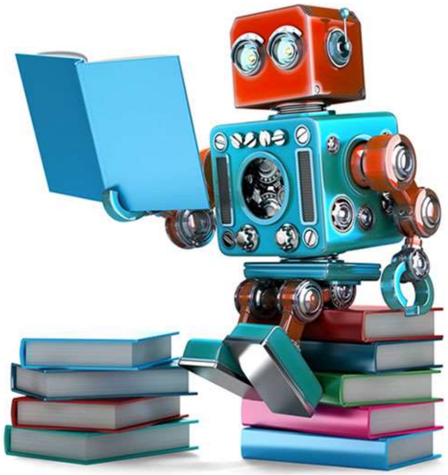
- Feigenbaum's universality conjecture (chaos theory)

  O.Lanford (1982)

- Kahler-Einstein metrics and other special metrics on complex projective varieties S.Donaldson (2005, ...)

Using ML to discover new mathematics:

- Finding counterexamples (disproving conjectures)
- Formulating new conjectures (learning new patterns)



Mathematics of AI: foundations, explainable AI, geometric deep learning, algorithm design, ...

Proof assistants: autoformalization, SAT-solvers, LEAN, Minerva, GPT-4, ...

Steve Smale

$n =$	1	2	3	4	5	6	7	8	9	10	11	12
TOP	1	1	1	1	1	1	1	1	1	1	1	1
PL	1	1	1	?	1	1	1	1	1	1	1	1
DIFF	1	1	1	?	1	1	28	2	8	6	992	1

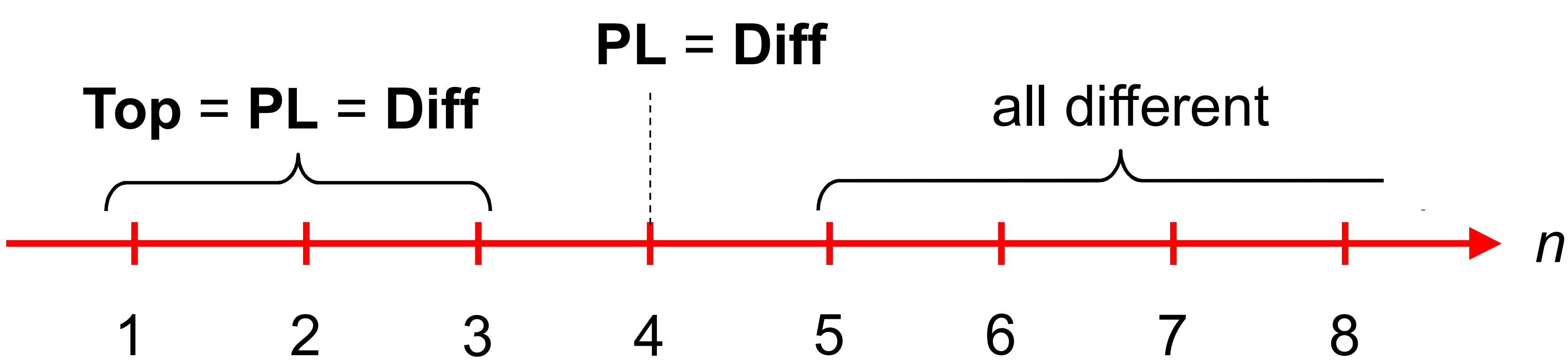
Number of homotopy *n*-spheres in each category.

John Milnor

The generalized Poincare conjecture:

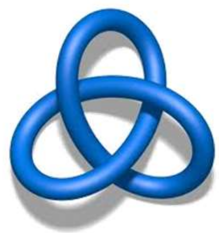
late 1950s

- **Top**: true for all *n*
- **PL**: true for all $n \neq 4$ ($n = 4$ currently not known)
- **Diff**: true for n = 1, 2, 3, 5, and 6



Generalized Poincare conjecture:

Every homotopy 4-sphere is diffeomorphic to the standard 4-sphere.



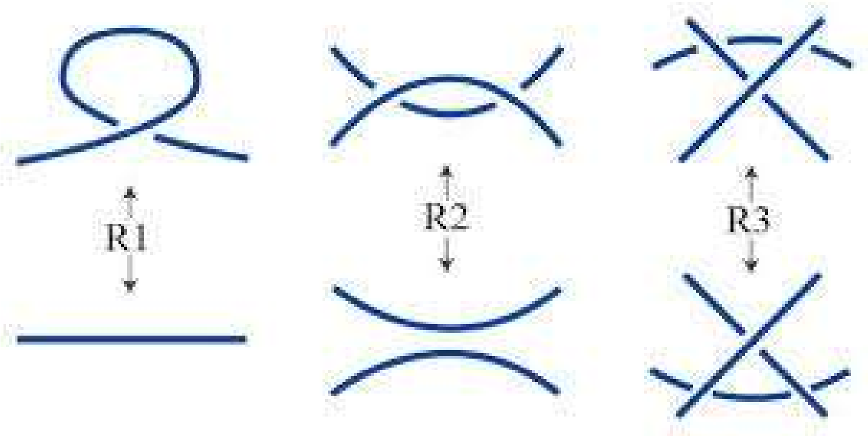
Theorem: If one finds a pair of knots which satisfy the following three properties:

- K and K′ have the same 0-surgery
- K is not slice
- K′ is slice

then the smooth 4-dimensional Poincare conjecture is false.

- Is it knotted?

S.G., J.Halverson, F.Ruehle, P.Sulkowski

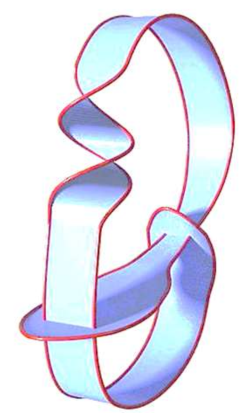


- Is it ribbon? Is it slice?

S.G., J.Halverson, C.Manolescu, F.Ruehle

(SPC4, slice-ribbon, ...)

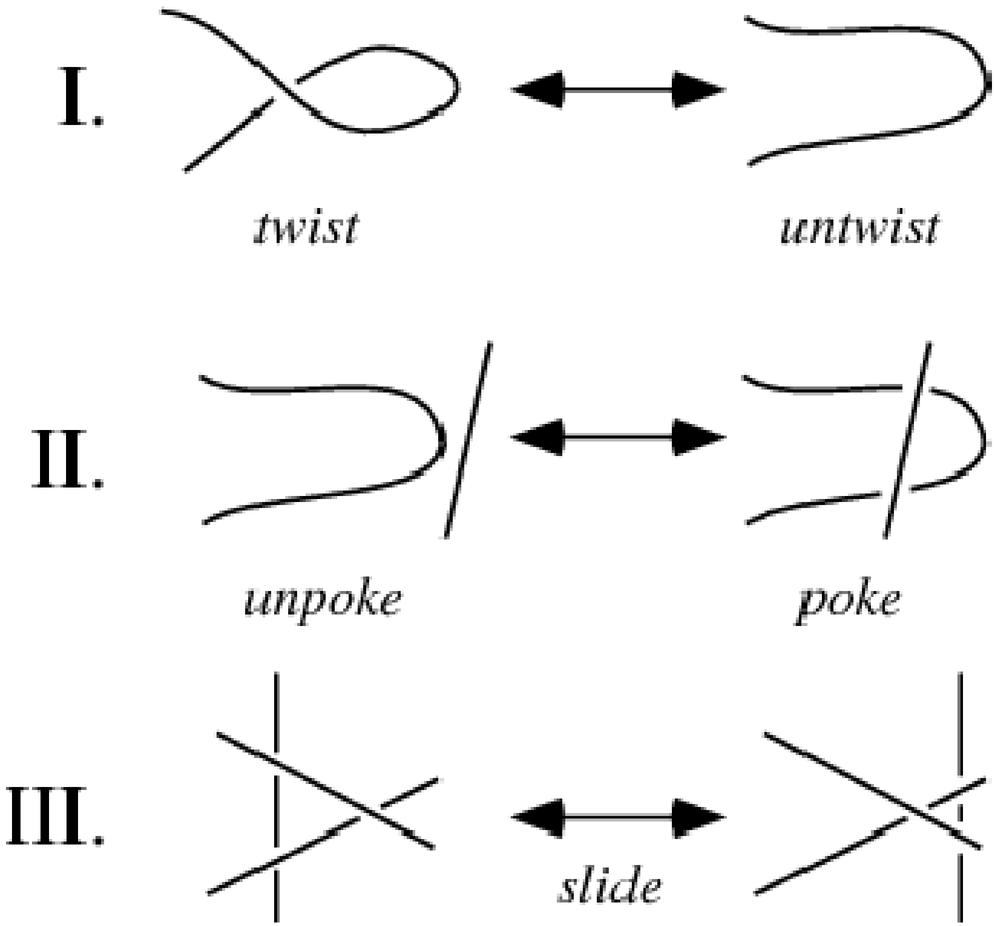
https://github.com/ruehlef/ribbon



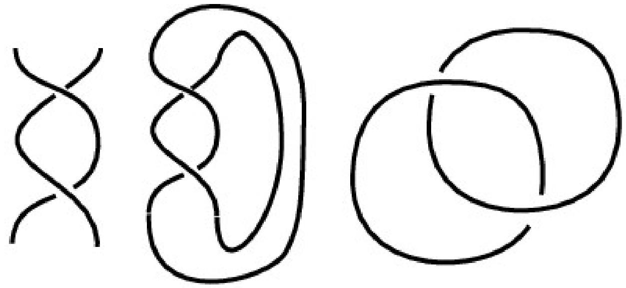
- Is it Andrews-Curtis trivial?

work in progress

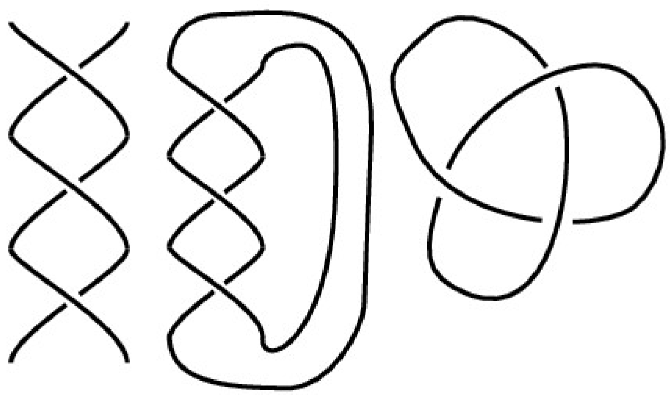
Combinatorial group theory



Kurt Reidemeister



Hopf Link



Trefoil Knot

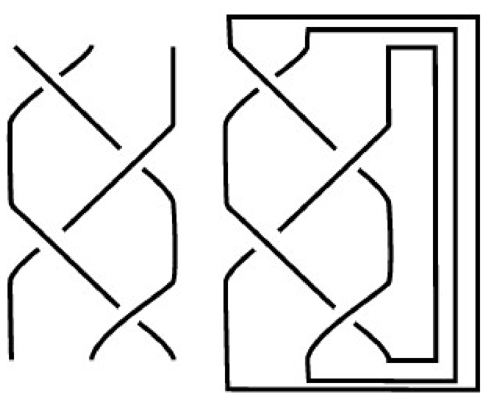
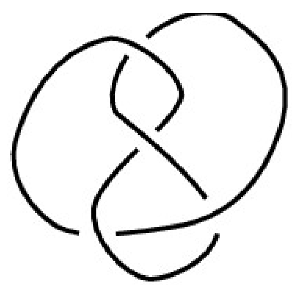
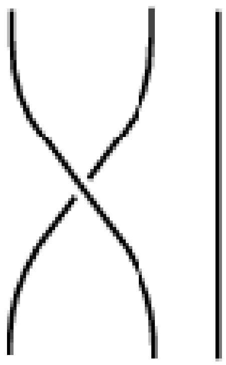
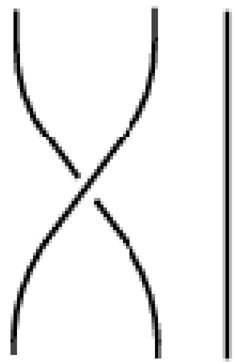



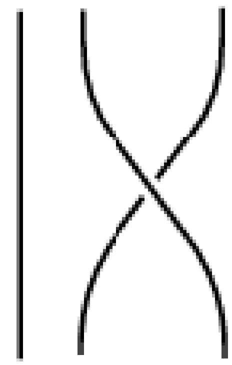
Figure Eight Knot



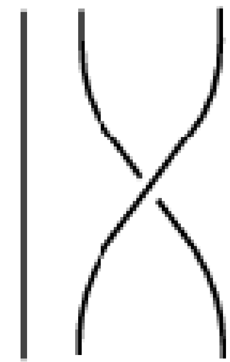
σ_1



σ_1^{-1}

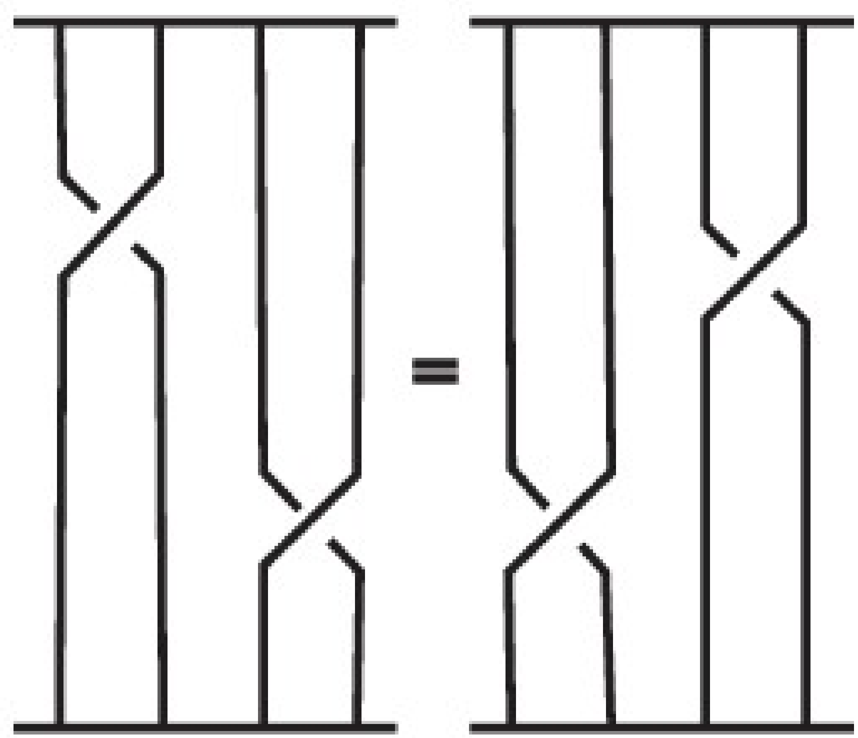


σ_2

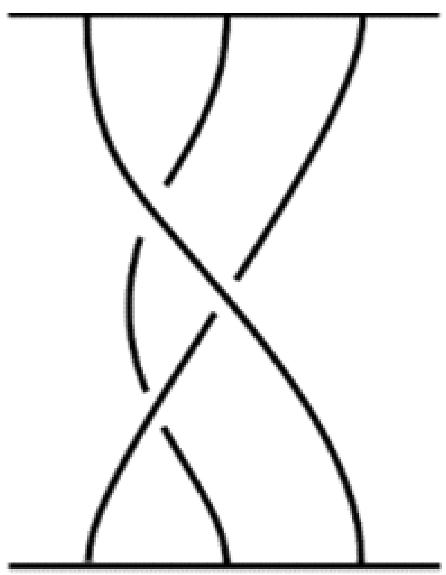


σ_2^{-1}

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{for } |i-j| > 1$$



$$\sigma_i \sigma_{i+1} \sigma_i \;=\; \sigma_{i+1} \sigma_i \sigma_{i+1}$$



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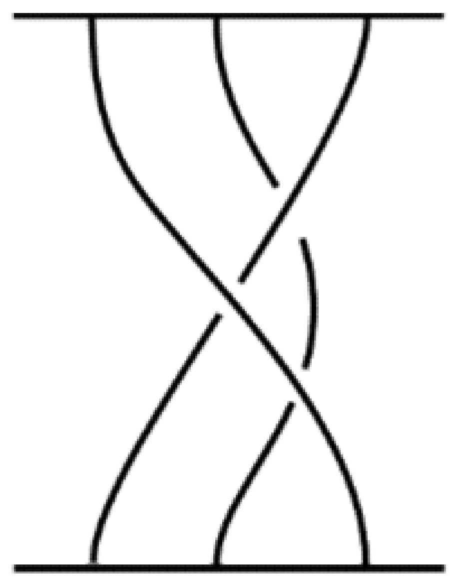
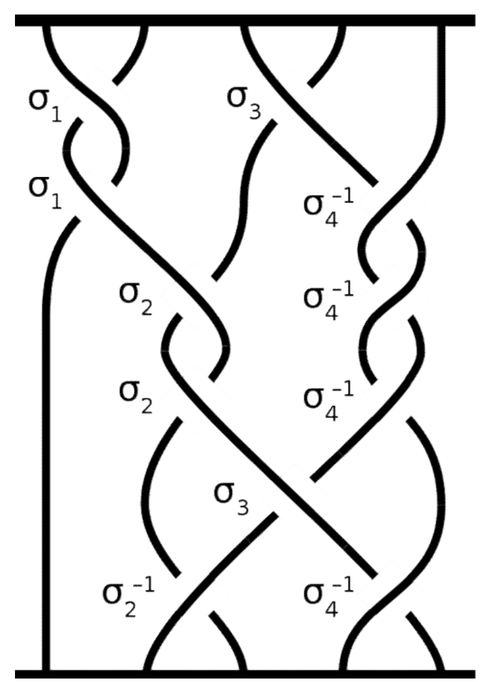
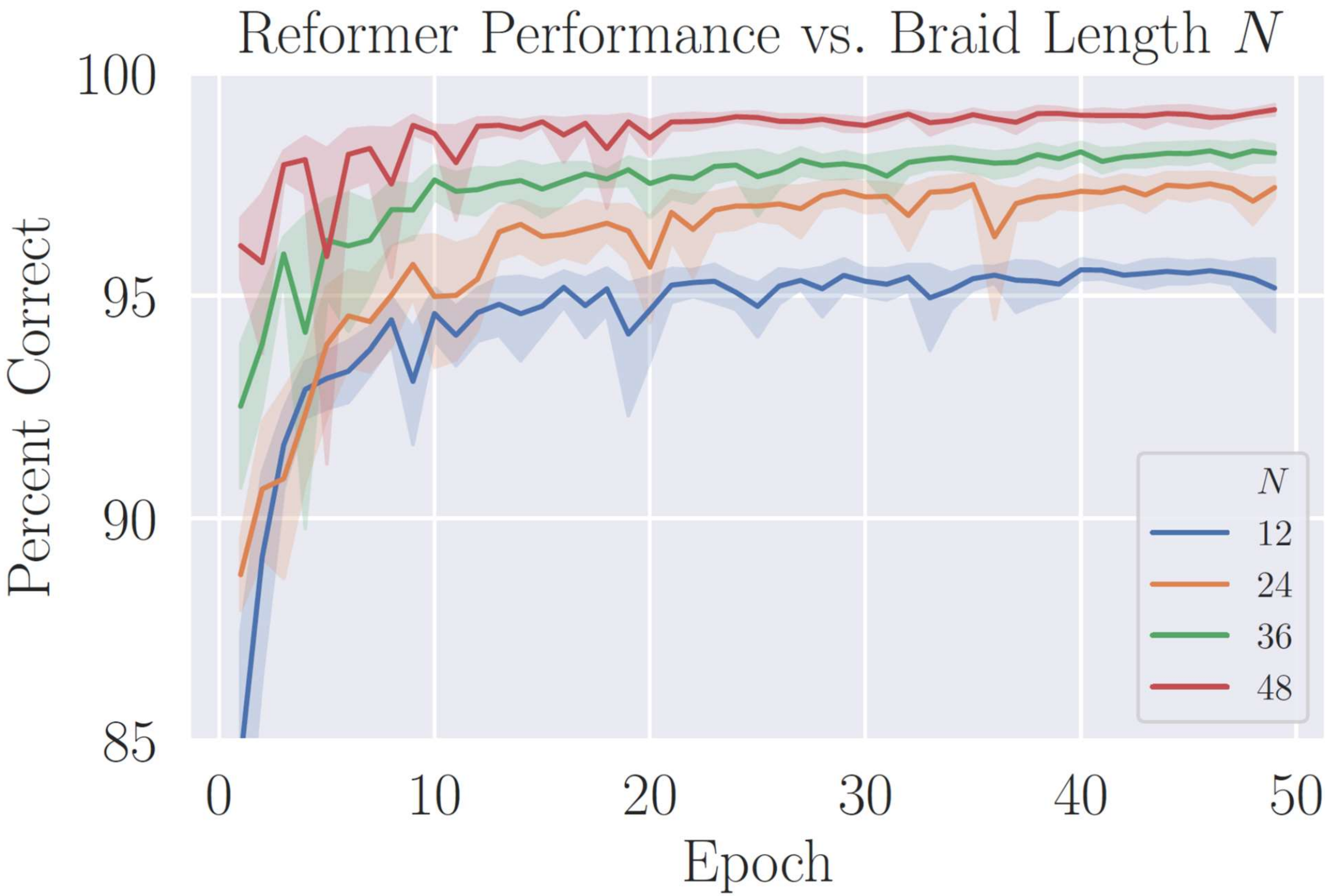


Learning to Unknot

Sergei Gukov[1], James Halverson[2,3], Fabian Ruehle[4,5], Piotr Sułkowski[1,6]



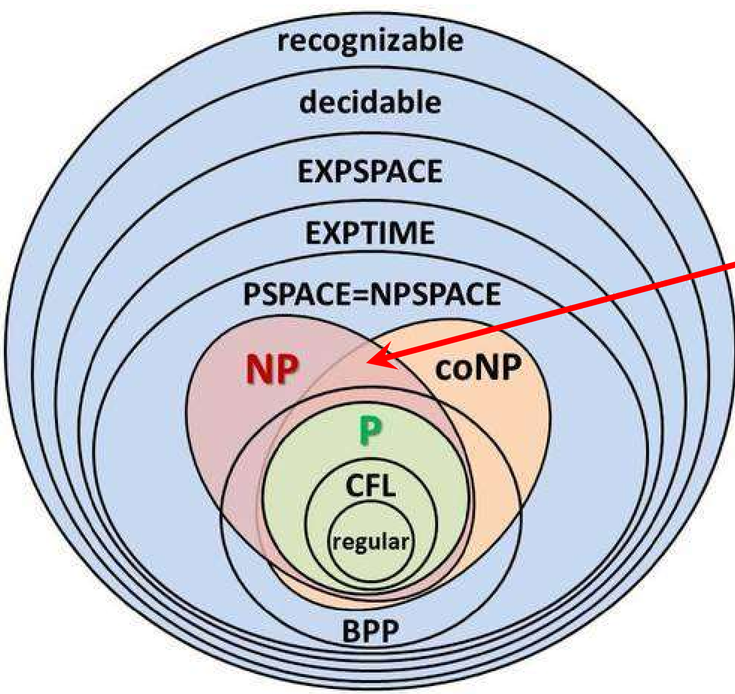


Reformer performance on UNKNOT as function of braid length. Performance increases with N.

Knottedness is in NP, modulo GRH

Greg Kuperberg*
Department of Mathematics, University of California, Davis, CA 95616

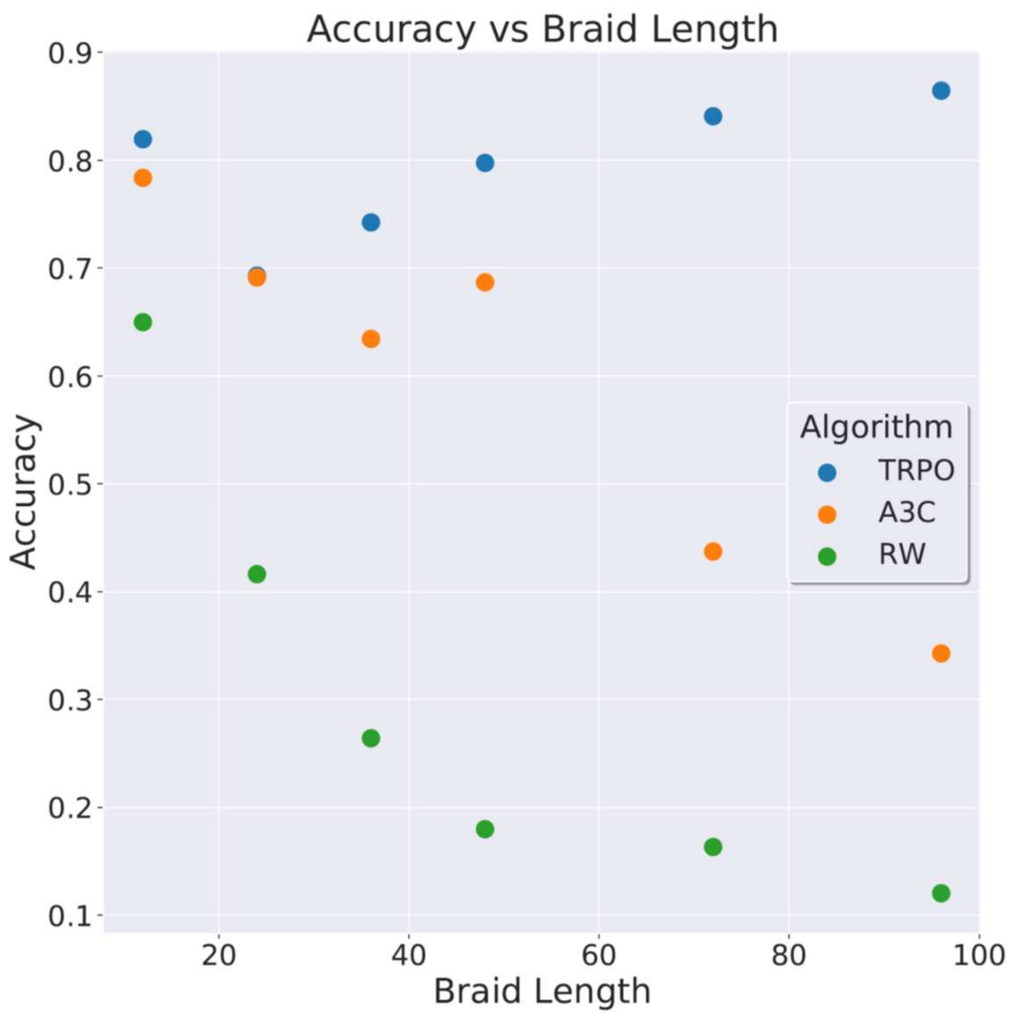
Given a tame knot K presented in the form of a knot diagram, we show that the problem of determining whether K is knotted is in the complexity class NP, assuming the generalized Riemann hypothesis (GRH). In other words, there exists a polynomial-length certificate that can be verified in polynomial time to prove that K is non-trivial. GRH is not needed to believe the certificate, but only to find a short certificate. This result complements the result of Hass, Lagarias, and Pippenger that unknottedness is in NP. Our proof is a corollary of major results of others in algebraic geometry and geometric topology.



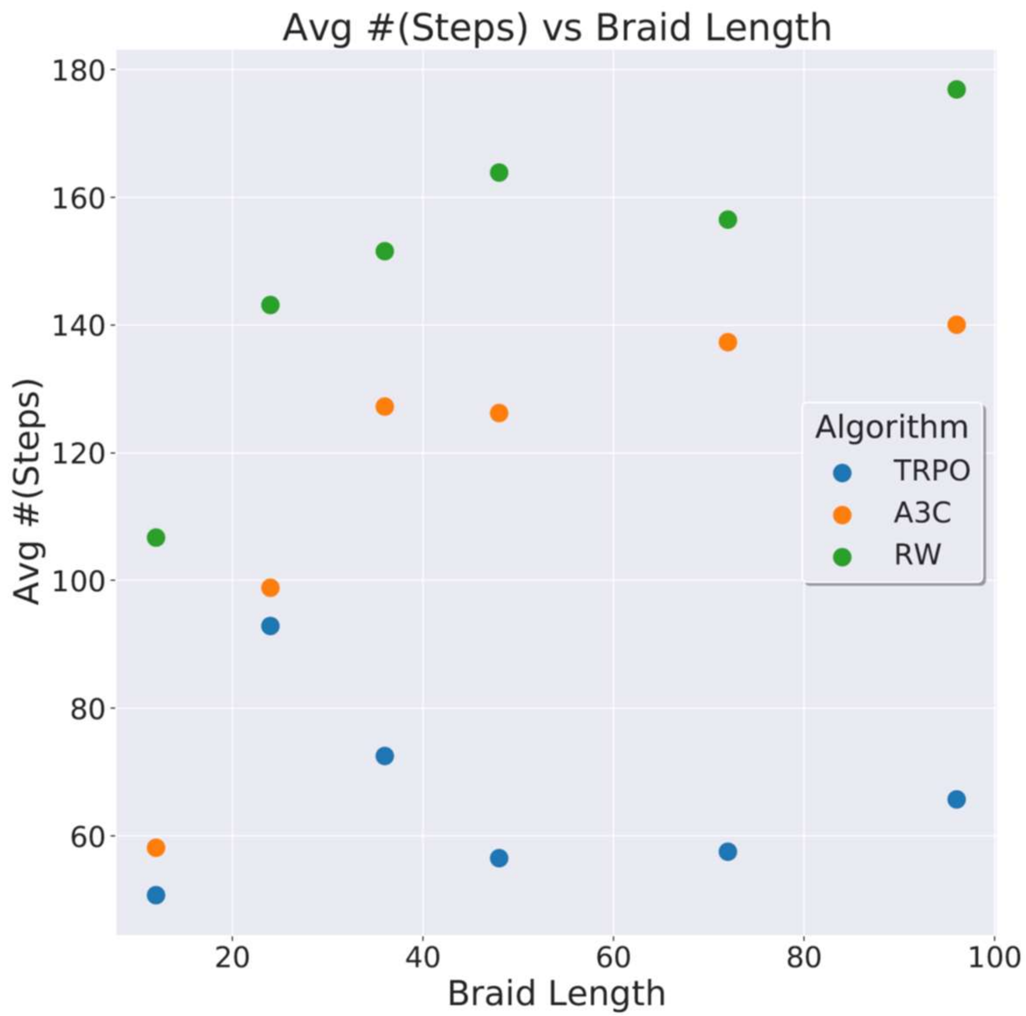
Unknottedness $\in$ NP $\cap$ coNP

integer $\overset{?}{=}$ product of two primes

:



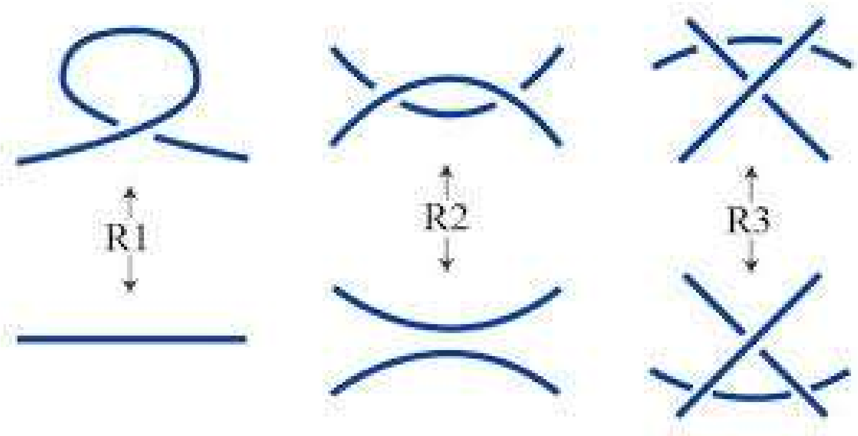
Fraction of unknots whose braid words could be reduced to the empty braid word as a function of initial braid word length.



Average number of actions necessary to reduce the input braid word to the empty braid word as a function of initial braid word length.

- Is it knotted?

S.G., J.Halverson, F.Ruehle, P.Sulkowski



- Is it ribbon? Is it slice?

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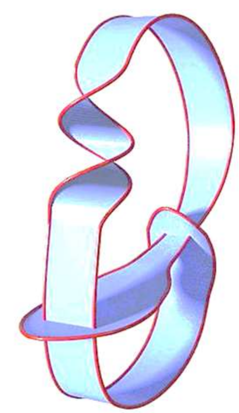
(SPC4, slice-ribbon, ...)

https://github.com/ruehlef/ribbon



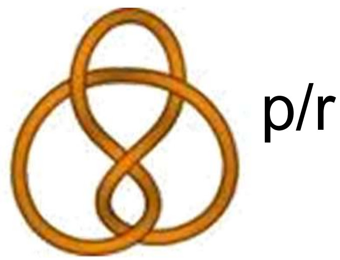
- Is it Andrews-Curtis trivial?

work in progress

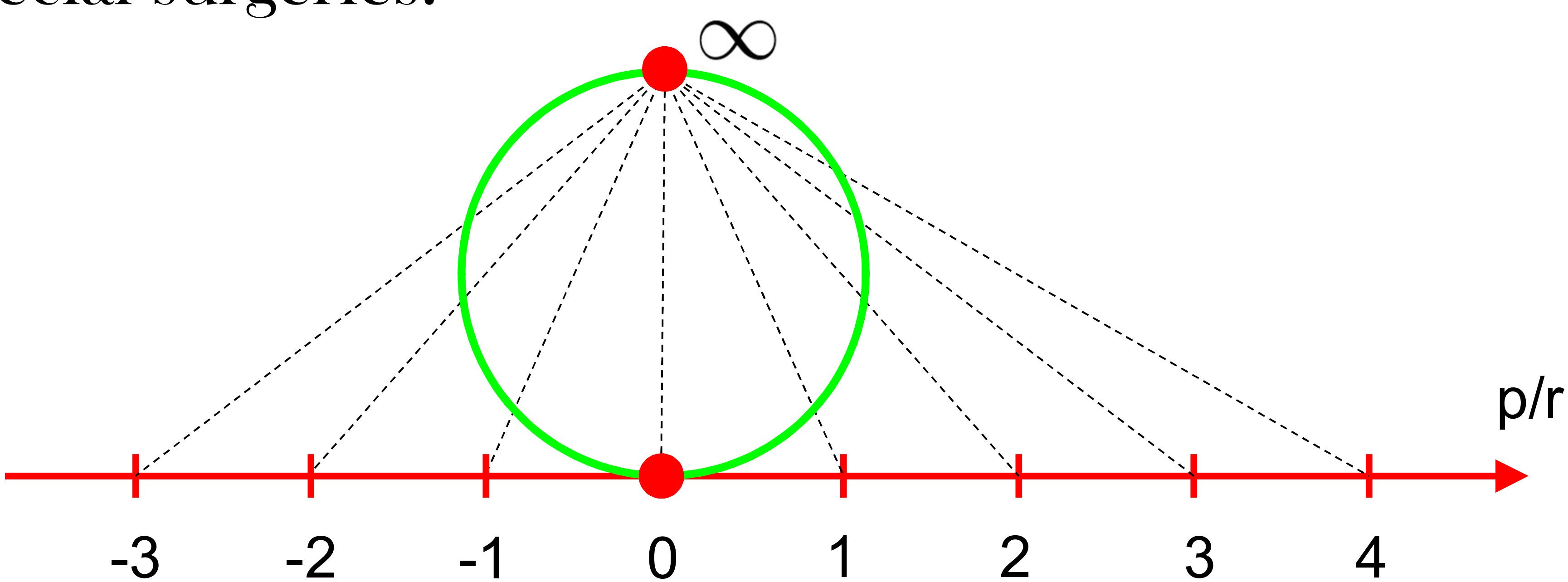
Combinatorial group theory

<u>Theorem [Lickorish, Wallace]:</u>

Every connected oriented closed 3-manifold arises by performing an integral Dehn surgery along a link in S^3.



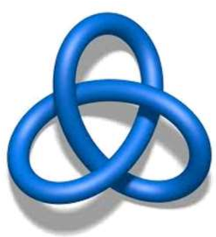
Special surgeries:



Property R

<u>Theorem ("property R" conjecture):</u> D.Gabai (1983)

If the 0-surgery on $K \subset S^3$ is homeomorphic to $S^1 \times S^2$, then K is the unknot.



The trefoil knot and the figure-8 knot are uniquely characterized by 0-surgery.

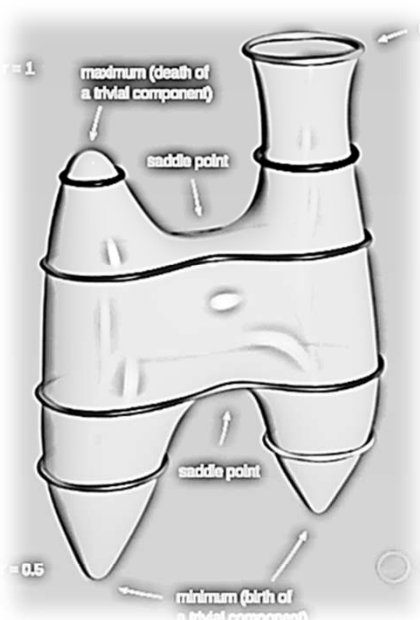
D.Gabai (1987)

$$M_3 = S^3_0(K)$$

FIBERED KNOTS AND POTENTIAL COUNTEREXAMPLES TO THE PROPERTY 2R AND SLICE-RIBBON CONJECTURES

ROBERT E. GOMPF, MARTIN SCHARLEMANN, AND ABIGAIL THOMPSON

48 crossings



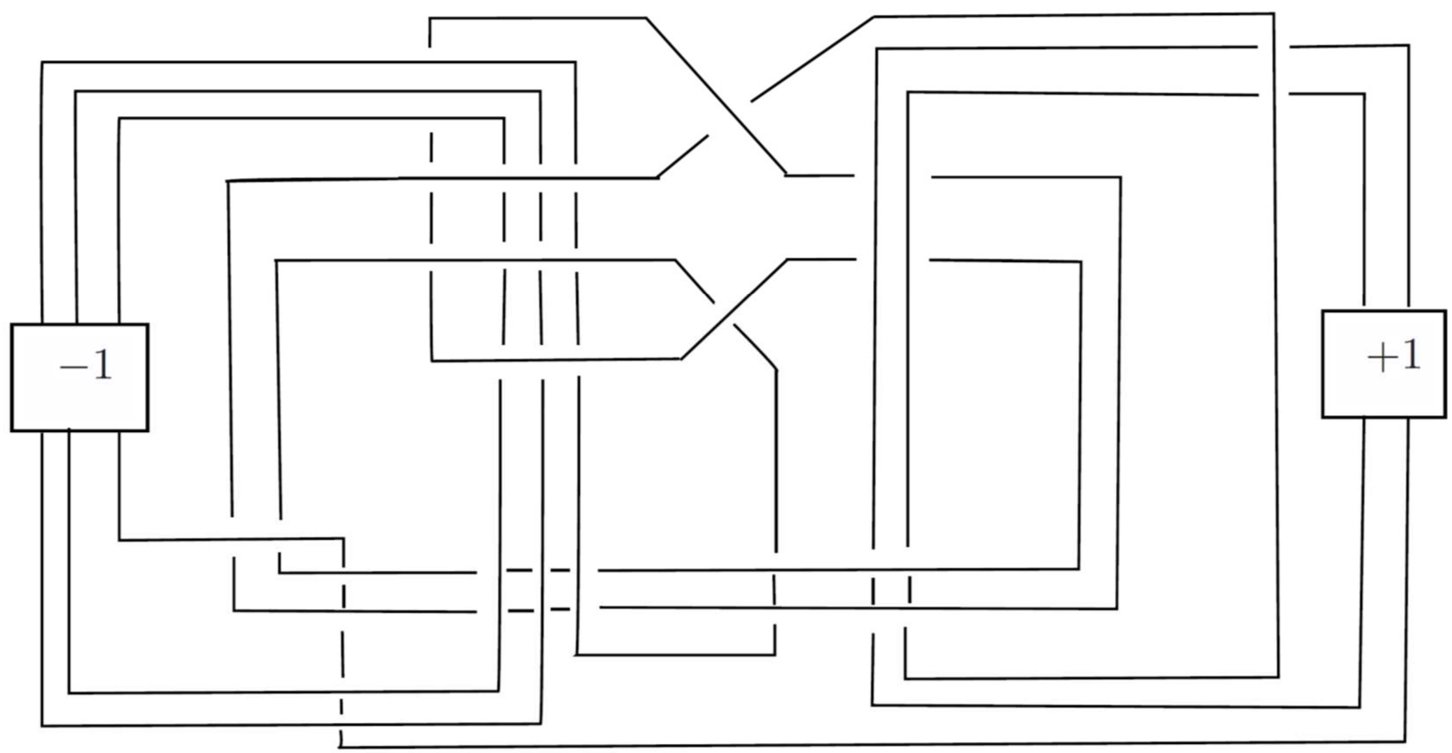
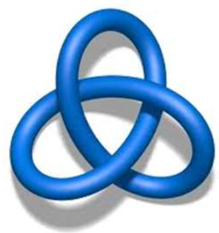


FIGURE 2. A slice knot that might not be ribbon

Generalized Poincare conjecture:

Every homotopy 4-sphere is diffeomorphic to the standard 4-sphere.



Theorem: If one finds a pair of knots which satisfy the following three properties:

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- K is not slice
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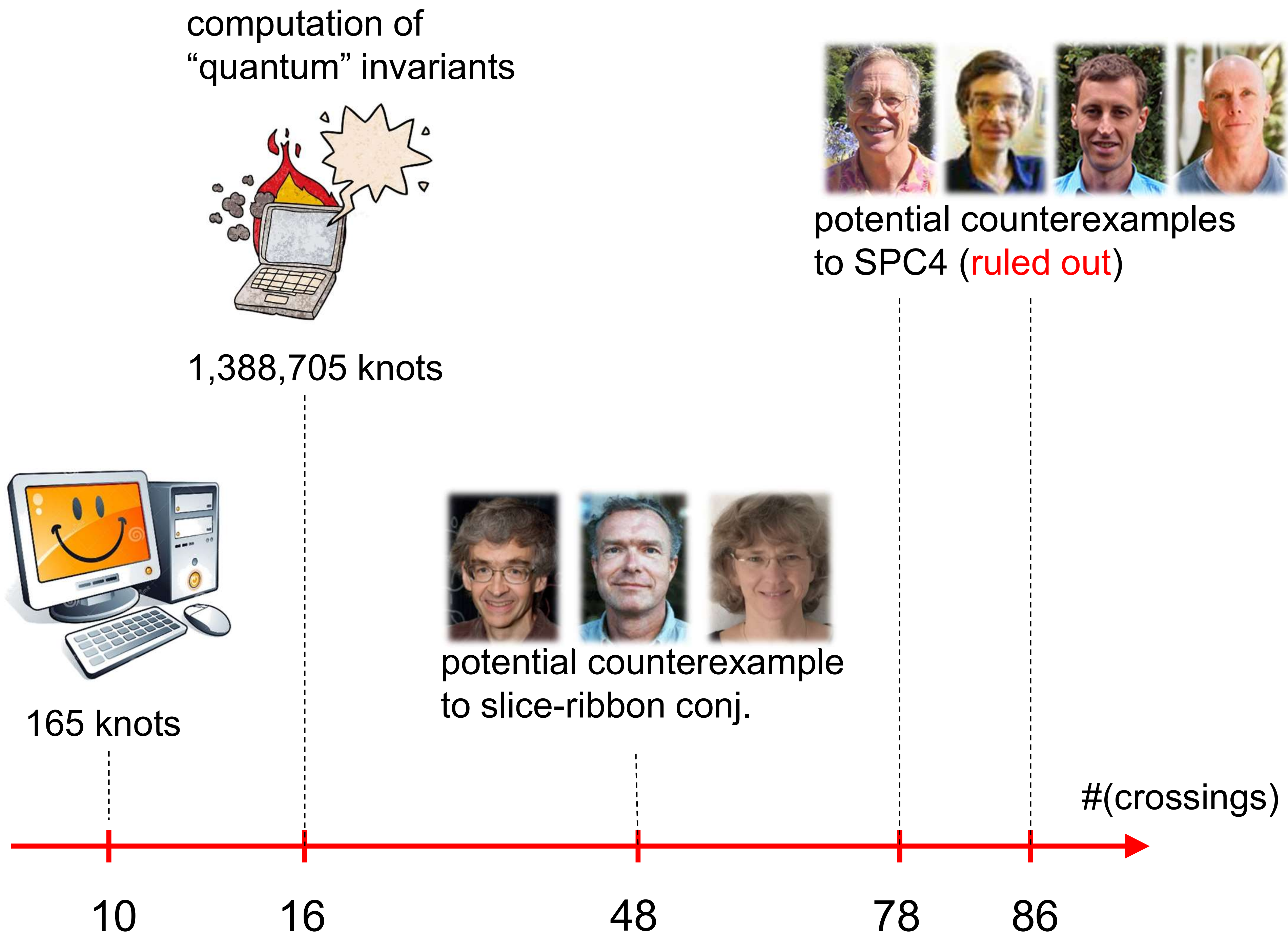
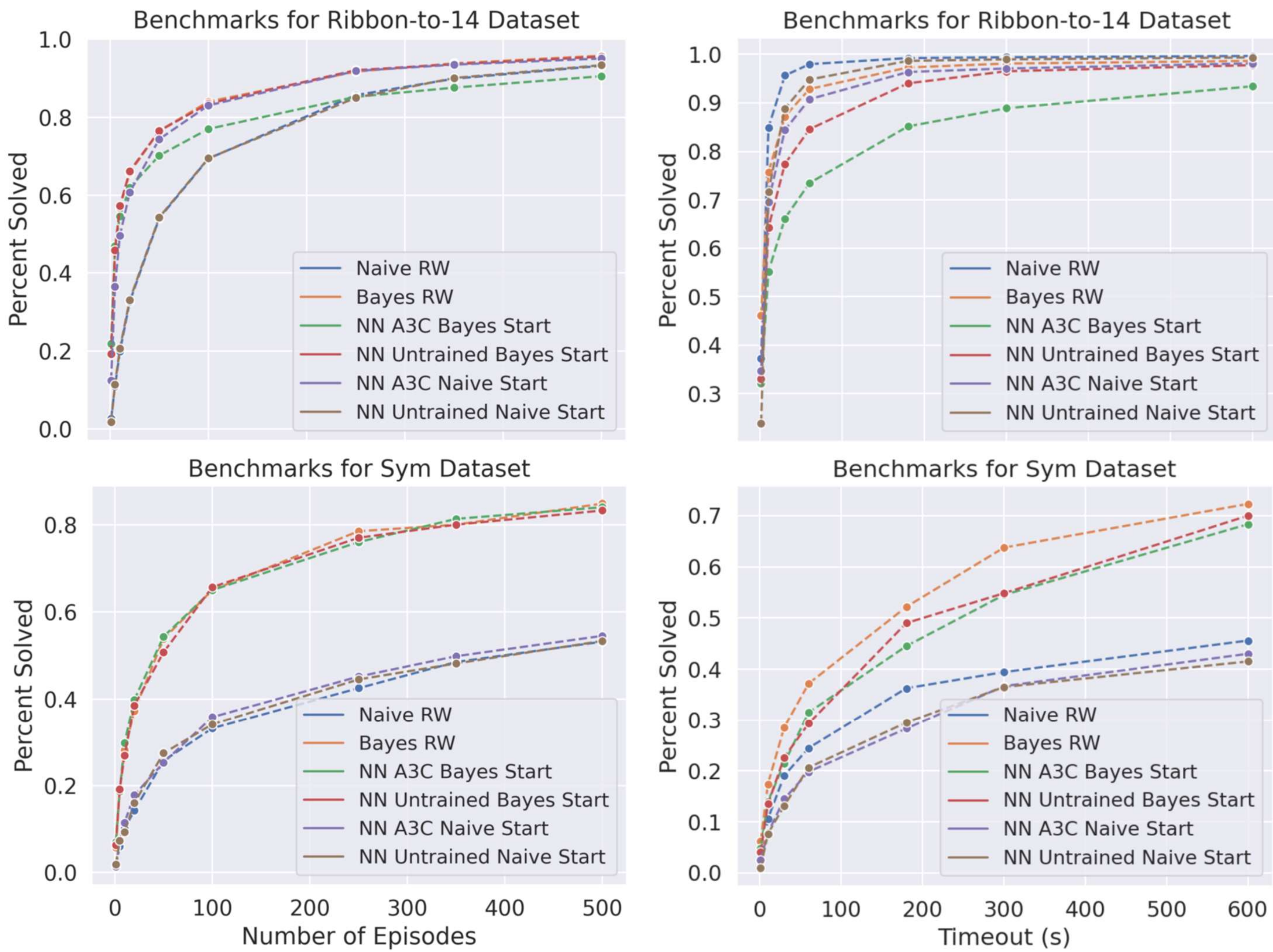
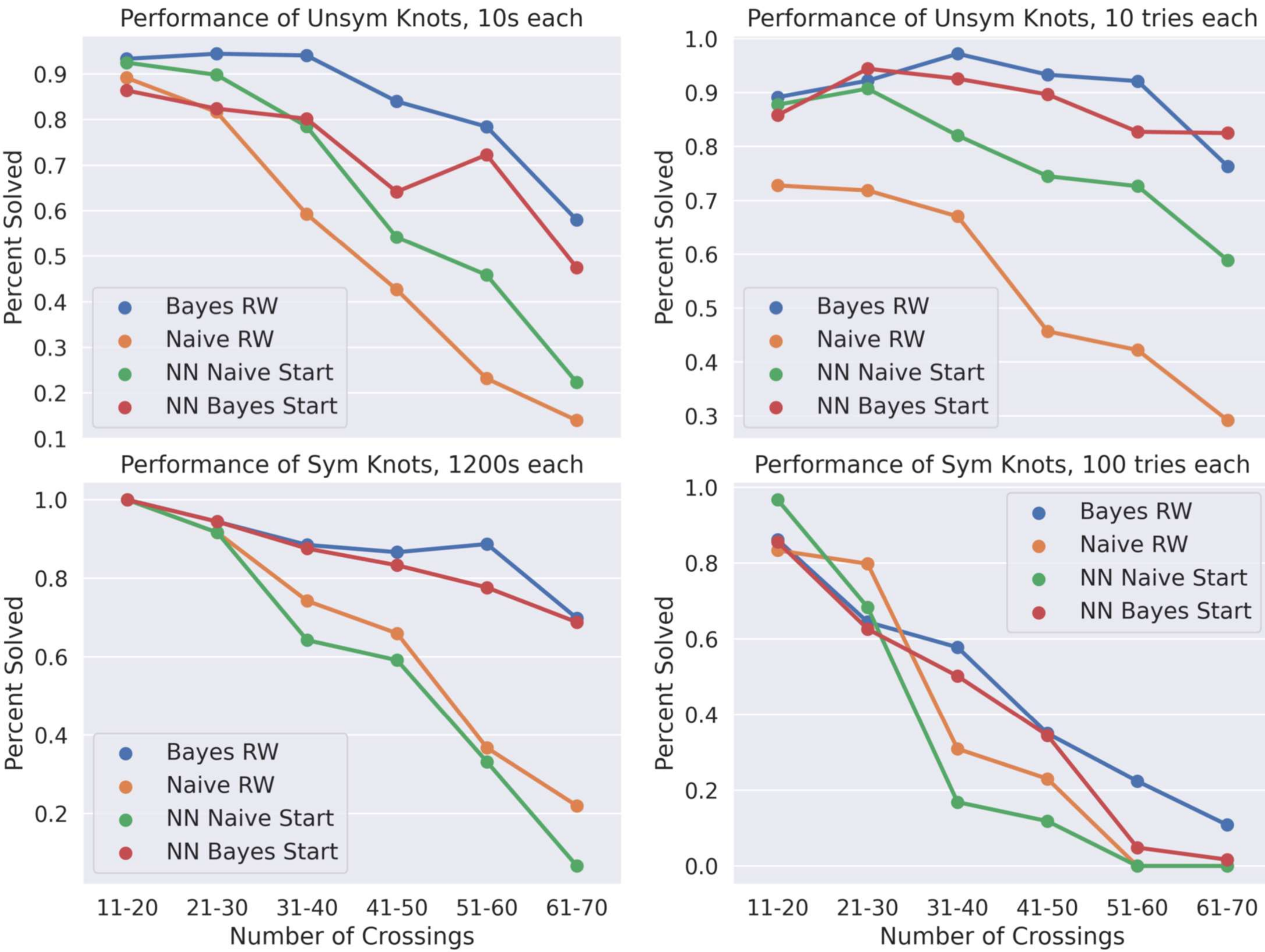
then the smooth 4-dimensional Poincare conjecture is false.


computation of
“quantum” invariants
1,388,705 knots
165 knots
potential counterexamples
to SPC4 (ruled out)
potential counterexample
to slice-ribbon conj.
#(crossings)
10
16
48
78
86


Benchmarks for Ribbon-to-14 Dataset
Benchmarks for Ribbon-to-14 Dataset
Benchmarks for Sym Dataset
Benchmarks for Sym Dataset
Percent Solved
Number of Episodes
Timeout (s)
Naive RW
Bayes RW
NN A3C Bayes Start
NN Untrained Bayes Start
NN A3C Naive Start
NN Untrained Naive Start
0.0
0.2
0.4
0.6
0.8
1.0
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0.1
0
100
200
300
400
500
600



Performance of Unsym Knots, 10s each
Performance of Unsym Knots, 10 tries each
Performance of Sym Knots, 1200s each
Performance of Sym Knots, 100 tries each
Percent Solved
Bayes RW
Naive RW
NN Naive Start
NN Bayes Start
11-20
21-30
31-40
41-50
51-60
61-70
Number of Crossings

Leveraging Procedural Generation to Benchmark Reinforcement Learning

Karl Cobbe [1] **Christopher Hesse** [1] **Jacob Hilton** [1] **John Schulman** [1]

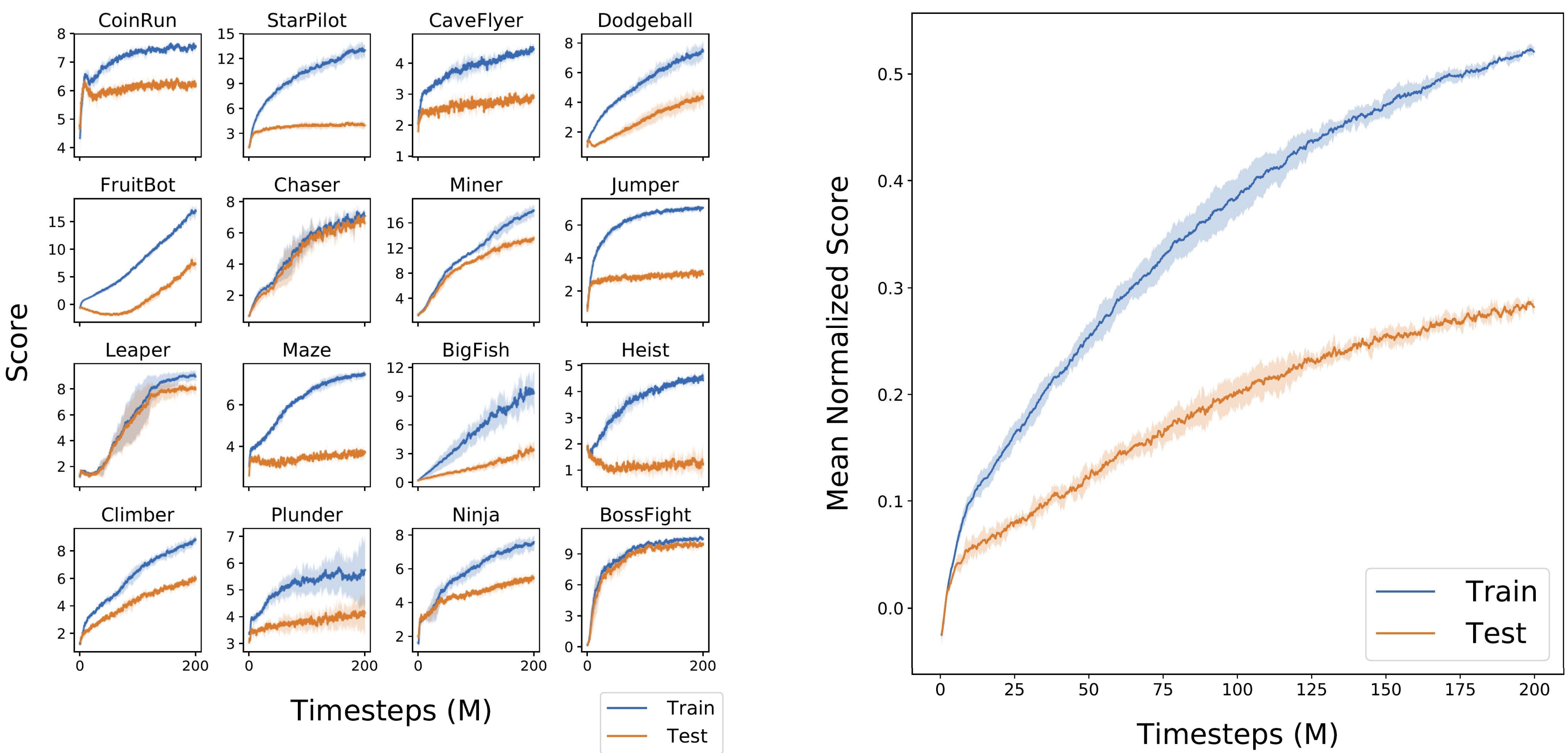
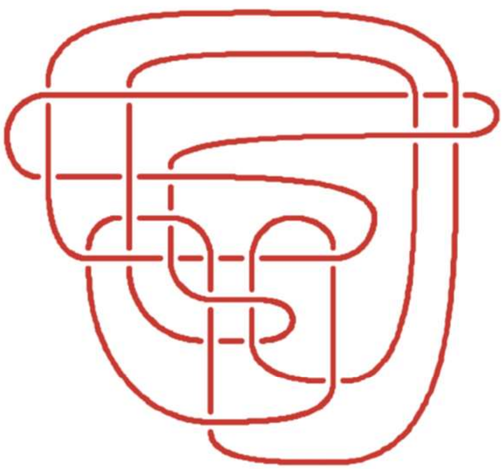
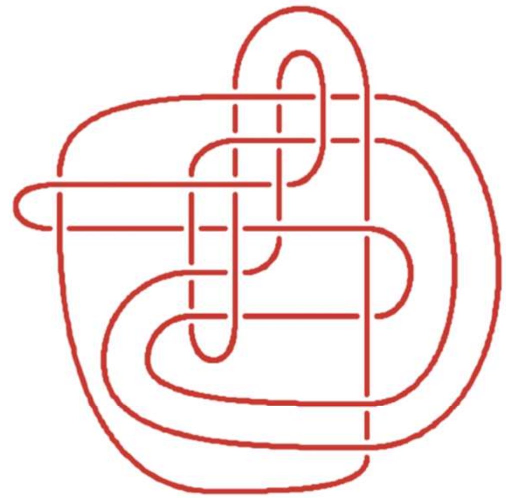


Figure 4. Generalization performance from 500 levels in each environment. The mean and standard deviation is shown across 3 seeds.

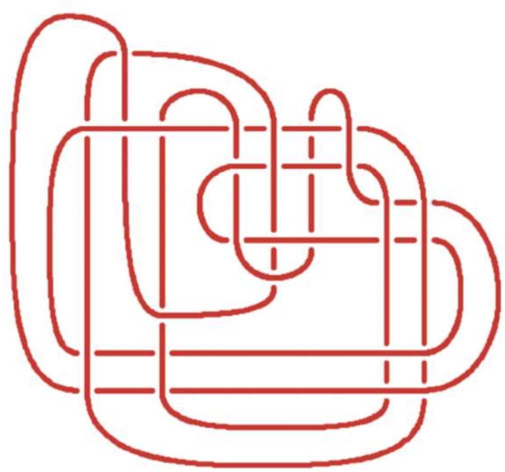
Could not find ribbons:



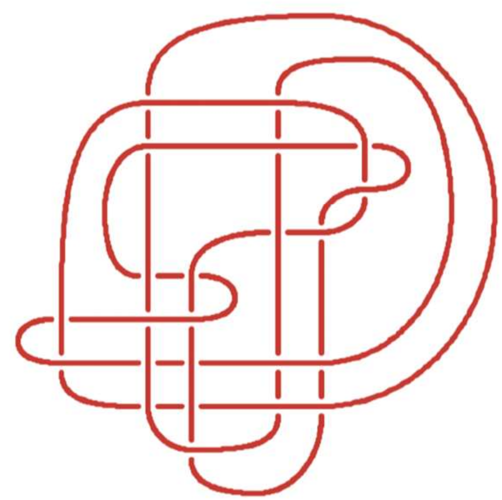
$K_G(0,1,-1,-1,1,0)$



$K_G(2,0,0,-1,2,-1)$

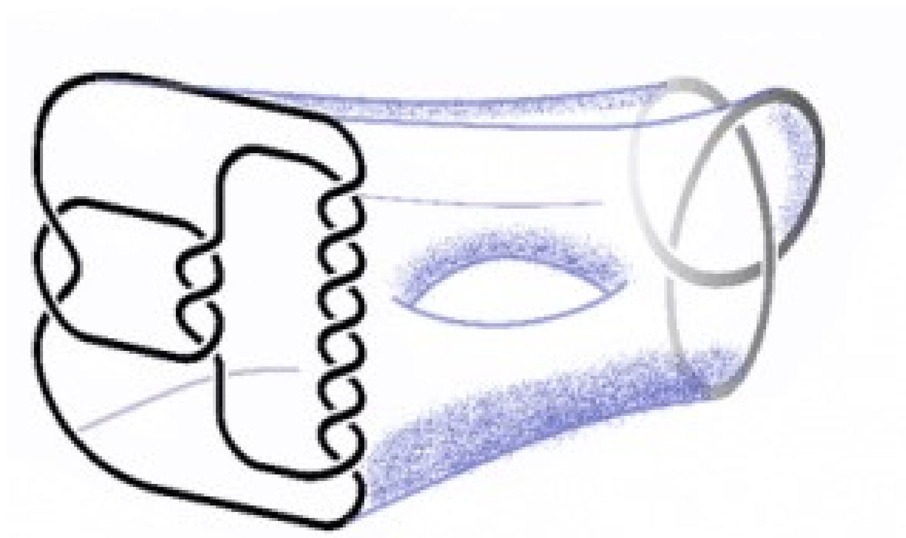


$K_B(0,1,2,0,-1,-1)$



$K_B(0,0,2,0,0,-1)$

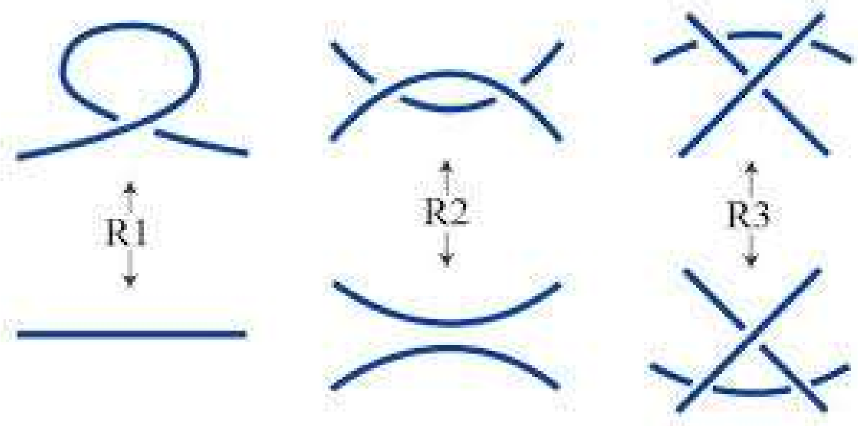
$$\{0\} = \left\{ \begin{matrix}\text{slice}\\ \text{knots}\end{matrix} \right\} \underbrace{\subset \ \dots \ \subset}_{\text{Cochran-Harvey-Horn}} \left\{\begin{matrix}\text{topologically}\\ \text{slice knots}\end{matrix}\right\} \underbrace{\subset \ \dots \ \subset}_{\text{Cochran-Orr-Teichner}} \overset{?}{\mathcal{C}}$$



Our computations indicate that K14a19470 is 2-torsion.

- Is it knotted?

S.G., J.Halverson, F.Ruehle, P.Sulkowski



Hard unknots

- Is it ribbon? Is it slice?

S.G., J.Halverson, C.Manolescu, F.Ruehle

(SPC4, slice-ribbon, ...)

https://github.com/ruehlef/ribbon

Hard ribbon knots

- Is it Andrews-Curtis trivial?

work in progress

Hard AC presentations

Conjecture [J.Andrews and M.Curtis '65]:

Every balanced presentation of the trivial group

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$$

can be reduced to the trivial presentation

$$\langle x_1, \ldots, x_n \mid x_1, \ldots, x_n \rangle$$

by a sequence of Andrews-Curtis (Nielsen) moves:

$$r_i,\, r_j \mapsto r_i r_j,\, r_j$$

$$r_i \mapsto r_i^{-1}$$

"handle slides"

$$r_i \mapsto x_j^{\pm 1} r_i x_j^{\mp 1}$$

"handle cancellation"

$$^{*}\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle \leftrightarrow$$

$$\leftrightarrow \langle x_1, \ldots, x_n, x_{n+1} \mid r_1, \ldots, r_n, x_{n+1} \rangle$$

---

* generalized

- No counterexamples with relations of total length <13
- Believed to be false
- Many potential counterexamples, e.g.

$$\langle\, x, y \mid xyx = yxy,\, x^{n+1} = y^n \,\rangle \quad n \geq 3$$

S.Akbulut, R.Kirby (1985)

- Validating any of these, disproves the following

Conjecture ("Generalized Property R"):

If surgery on an n-component link L yields the connected sum $(S^1 \times S^2)^{\#n}$, then L is obtained from the 0-framed unlink by a sequence of handle slides.

R.Gompf, M.Scharlemann, A.Thompson (2010)

- A handle decomposition of a homotopy sphere without 3-handles gives a balanced presentation of the trivial group

- AC moves = Kirby moves (without introducing 3-handles)

$$\Downarrow$$

- A potential counterexample to AC gives a potential counterexample to SPC4

<u>Theorem:</u>

R.Gompf (1991)

$$\langle\, x, y \mid xyx = yxy,\ x^5 = y^4 \,\rangle$$

gives a standard 4-sphere.

International Journal of Algebra and Computation
Vol. 13, No. 1 (2003) 61–68

BREADTH-FIRST SEARCH AND THE ANDREWS–CURTIS CONJECTURE

GEORGE HAVAS* and COLIN RAMSAY†

Centre for Discrete Mathematics and Computing
School of Information Technology and Electrical Engineering
The University of Queensland, Queensland 4072, Australia

Theorem. *Let G be a group defined by a balanced presentation on two generators, with the sum of the relator lengths at most thirteen. Then:*

(i) *if G has trivial abelianization, G is trivial or is isomorphic to $L_2(5)$, the unique perfect group of order* 120;
(ii) *if G is trivial, its presentation is AC-equivalent to the standard presentation or to the presentation* $\langle x, y \mid x^3 = y^4, xyx = yxy \rangle$.

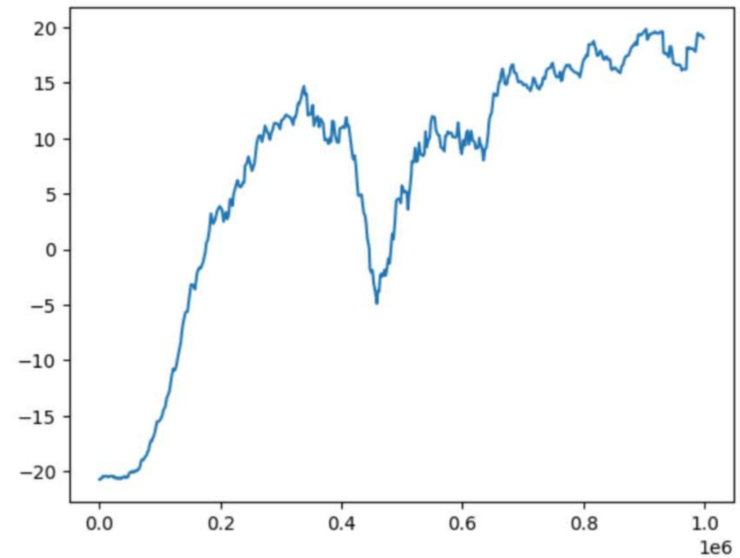
T\L	10	11	12	13	14	15	16	17	18	19	20
13	**4**	**4**	**4**	**4**	**4**	**4**	**4**	**4**	**4**	**4**	**4**
14	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**
15	**70**	**70**	**70**	**70**	**70**	**70**	**70**	**70**	**70**	**70**	**70**
16	64	**86**	**86**	**86**	**86**	**86**	**86**	**86**	**86**	**86**	**86**
17	220	416	454	**458**	**458**	**458**	**458**	**458**	**458**	**458**	**458**
18	98	392	398	**590**	**590**	**590**	**590**	**590**	**590**	**590**	**590**
19	240	764	1382	2854	**3226**	**3226**	**3226**	**3226**	**3226**	**3226**	**3226**
20	10	442	522	2004	2082	3352	3352	**3356**	**3356**	**3356**	**3356**
21	20	746	1624	3870	8334	16948	19666	19690	19690	**19692**	**19692**
22	0	438	570	2812	3714	12288	12584	23174	23174	23188	23192
23	0	112	1462	4474	9194	21678	41492	101544	128356	128380	128388
24	0	6	42	3400	3858	12978	15458	61100	64686	150264	150276
25	0	0	110	4350	11246	22422	42550	102262	236860	631000	843778
26	0	0	0	4306	5384	17930	19668	62874	83902	375818	394172
27	0	0	0	710	13548	28176	51590	96714	196098	538380	1269016
28	0	0	0	52	494	26008	27874	76930	83864	289920	364040
29	0	0	0	0	1652	30934	77162	123178	230774	445036	953378
30	0	0	0	0	2	20430	24146	128556	138478	355754	405746
31	0	0	0	0	0	5854	62178	159086	368336	546680	1041462
32	0	0	0	0	0	326	3338	122164	130302	597064	639362
33	0	0	0	0	0	0	6314	151550	353810	730650	1758270
34	0	0	0	0	0	0	62	128556	150518	538278	585132
35	0	0	0	0	0	0	0	22772	374246	872784	1519374
36	0	0	0	0	0	0	0	1848	19030	762768	813708
37	0	0	0	0	0	0	0	0	51496	1016332	2112918
38	0	0	0	0	0	0	0	0	522	848998	946260
39	0	0	0	0	0	0	0	0	0	209668	2414958
40	0	0	0	0	0	0	0	0	0	19332	120852
41	0	0	0	0	0	0	0	0	0	0	270942
42	0	0	0	0	0	0	0	0	0	0	12062

TABLE 1. Each cell shows the number of pairs AC-equivalent to AK(3) of total length T obtained by the program when run with the length bound L. Highlighted cells do not increase when L is increased.

D.Panteleev, A.Ushakov

First steps

1.<x, y^{-1}>
2.<x^{-1}, x y x^{-1}>
3.<x x y^{-1} x^{-1}, x y x^{-1}>
4.<x x y^{-1} x^{-1}, x y^{-1} x^{-1}>
5.<x x y^{-1} x^{-1}, y x y^{-1} x^{-1} y^{-1}>
6.<x x y x^{-1}, yxyx^{-1}y^{-1}>



OpenAI Baselines implementations

Example	PPO	A2C	A3C	DQN
1	00:04:51	00:01:08	00:00:43	00:02:39
2	00:07:19	00:01:53	00:02:50	00:02:52
3	00:08:05	Terminated (15 mins)	00:04:47	Terminated (15 mins)
4	00:13:00	--	00:11:50	--
5	00:14:17	--	00:13:00	--
6	00:13:50	--	Terminated (30 mins)	--

THE COMPLEXITY OF BALANCED PRESENTATIONS AND THE ANDREWS–CURTIS CONJECTURE

MARTIN R. BRIDSON

Hard AC presentations

Theorem A. *For $k \geq 4$ one can construct explicit sequences of k-generator balanced presentations $\mathcal{P}_n$ of the trivial group so that*

(1) the presentations $\mathcal{P}_n$ are AC-trivialisable;
(2) the sum of the lengths of the relators in $\mathcal{P}_n$ is at most $24(n+1)$;
(3) the number of (dihedral) AC moves required to trivialise $\mathcal{P}_n$ is bounded below by the function $\Delta(\lfloor \log_2 n \rfloor)$ where $\Delta : \mathbb{N} \to \mathbb{N}$ is defined recursively by $\Delta(0) = 2$ and $\Delta(m+1) = 2^{\Delta(m)}$.

7.4. **An Example.** Let me close by writing down an explicit presentation to emphasize that the explosive growth in the length of AC-trivialisations begins with relatively small presentations. Here is a balanced presentation of the trivial group that requires more than 10^{10000} AC-moves to trivialise it. We use the commutator convention $[x, y] = xyx^{-1}y^{-1}$.

$$\langle a, t, \alpha, \tau \mid [tat^{-1}, a]a^{-1}, \quad [\tau\alpha\tau^{-1}, \alpha]\alpha^{-1},$$
$$\alpha t^{-1}\alpha^{-1}[a,\ [t[t[ta^{20}t^{-1},\ a]t^{-1},\ a]t^{-1},\ a]],$$
$$a\tau^{-1}a^{-1}[\alpha,\ [\tau[\tau[\tau\alpha^{20}\tau^{-1},\ \alpha]\tau^{-1},\ \alpha]\tau^{-1},\ \alpha]]\rangle.$$

Mark your calendar!

Dec. 10-13: Mathematics and ML

https://mathml2023.caltech.edu/

Dec. 13-16: String Data 2023

https://stringdata2023.caltech.edu/

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