LeanDojo: Theorem Proving with Retrieval-Augmented Language Models

Kaiyu Yang

Postdoc @ Computing + Mathematical Sciences

Caltech

Theorem Proving in Proof Assistants

Proof assistant

Theorem Proving in Proof Assistants

Human

```
theorem gcd_self (n : nat) : gcd n n = n :=
```

Proof assistant

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$n : \mathbb{N}$
$\vdash \text{gcd } n\ n = n$

Proof assistant

Theorem Proving in Proof Assistants

Human

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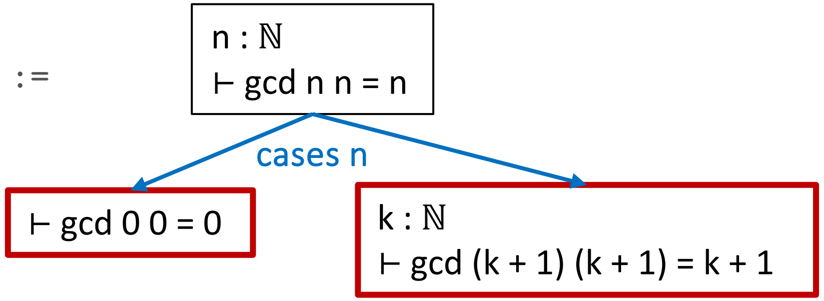
  cases n,
```



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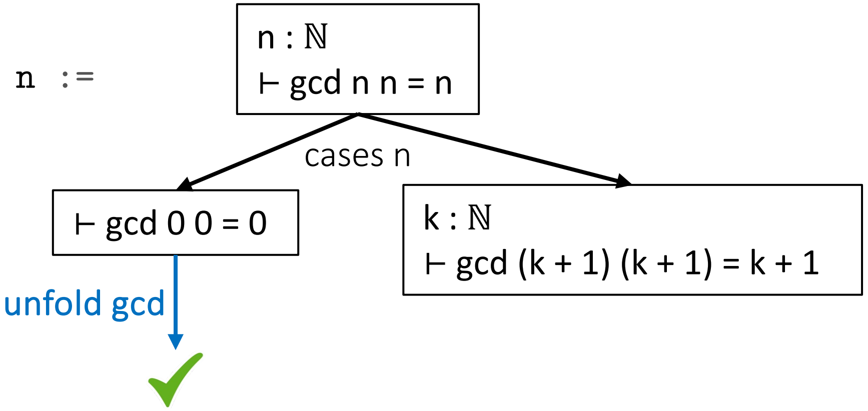

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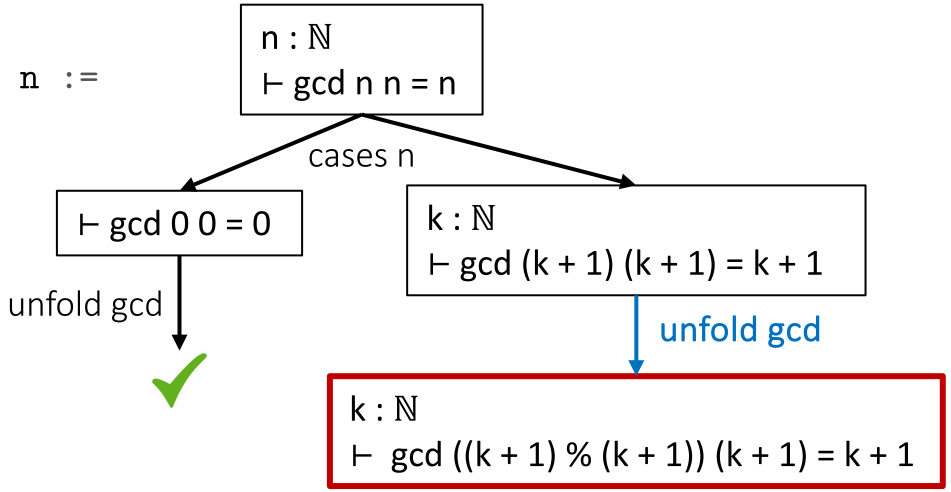

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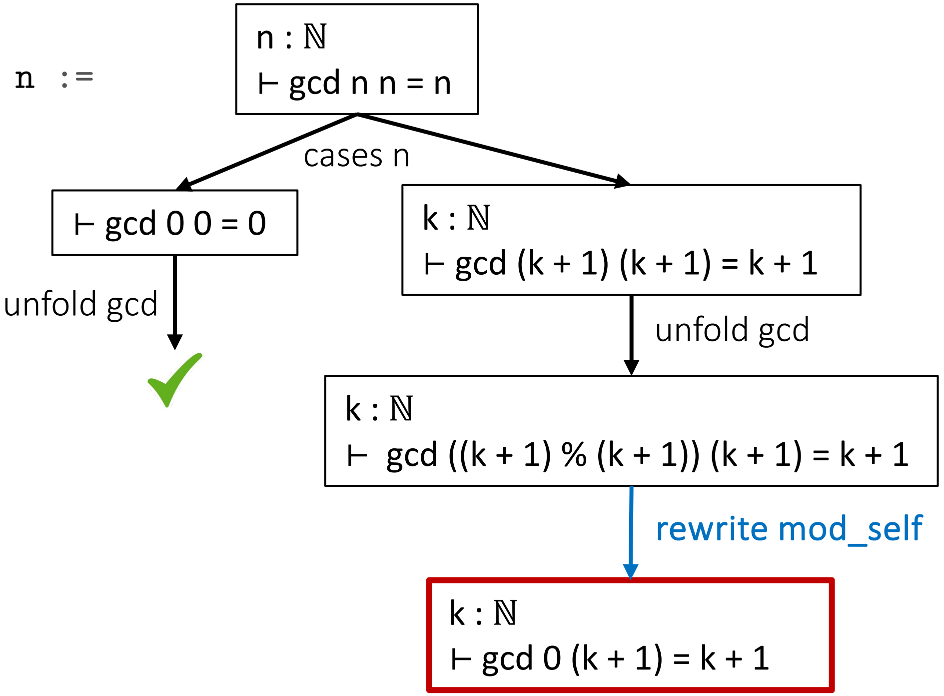

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Theorem Proving in Proof Assistants

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theorem gcd_self (n : nat) : gcd n n = n :=
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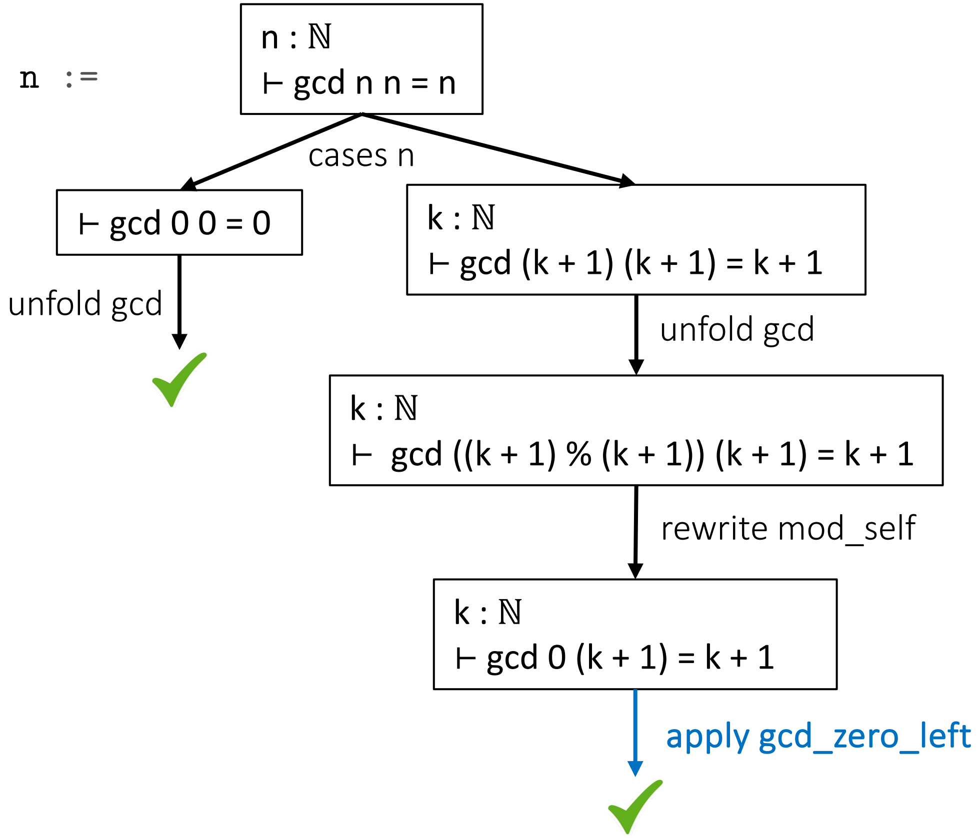

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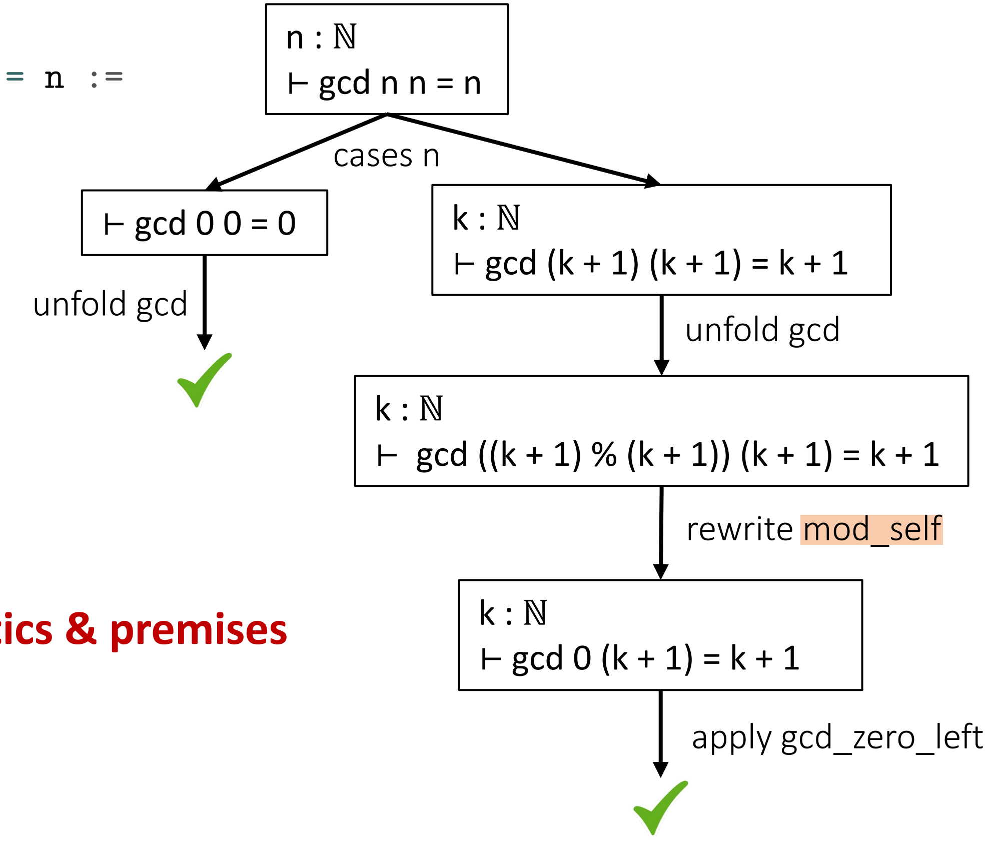
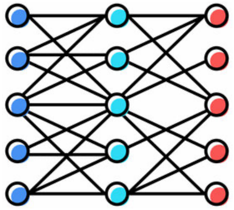


Proof assistant

- **Bottleneck: Finding the right tactics & premises**

Theorem Proving in Proof Assistants

Human



Machine learning

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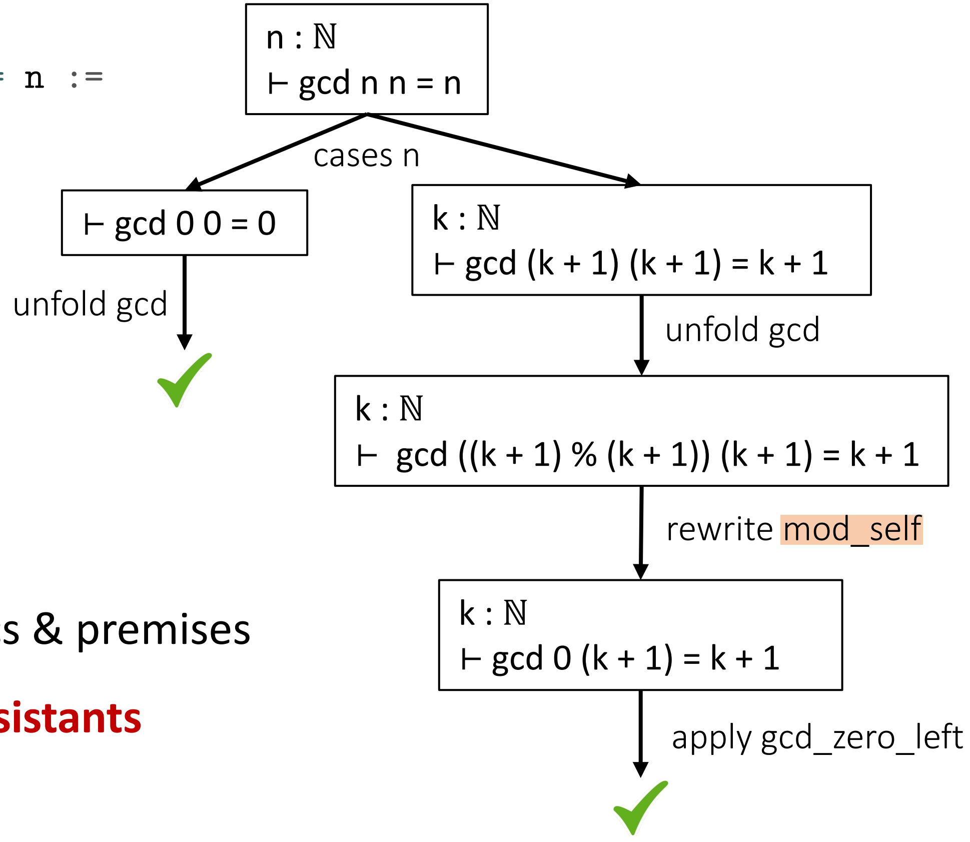
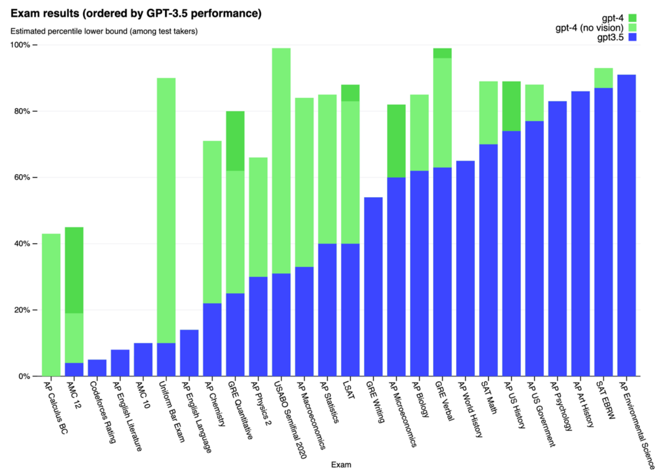


Proof assistant

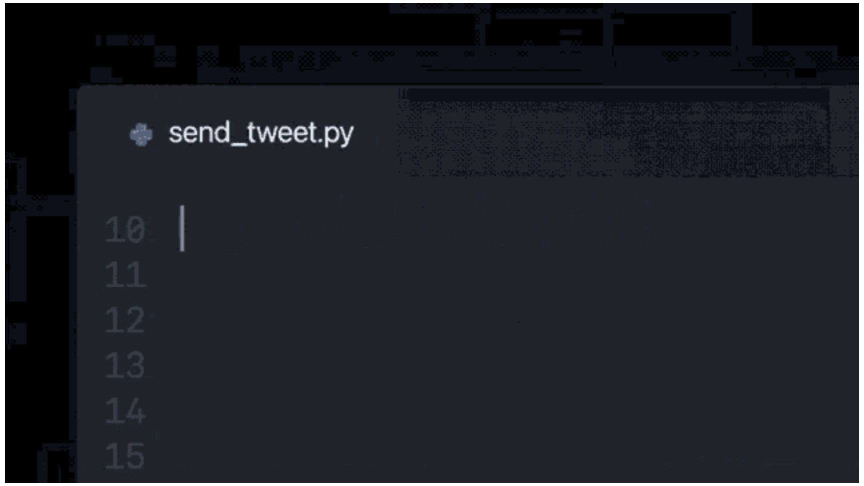
- Bottleneck: Finding the right tactics & premises
- **Learning to interact with proof assistants**

Large Language Models (LLMs)

- Very big neural networks, massive data, predicting the next word
- Good at elementary math and coding



GPT-4 on standard exams (SAT, LSAT, etc.)



GitHub Copilot

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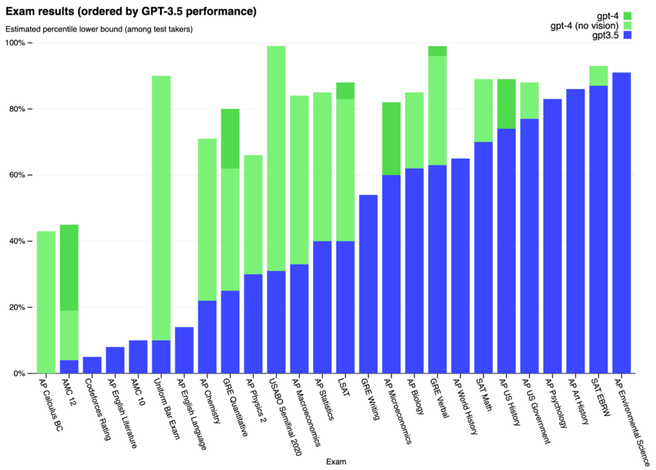

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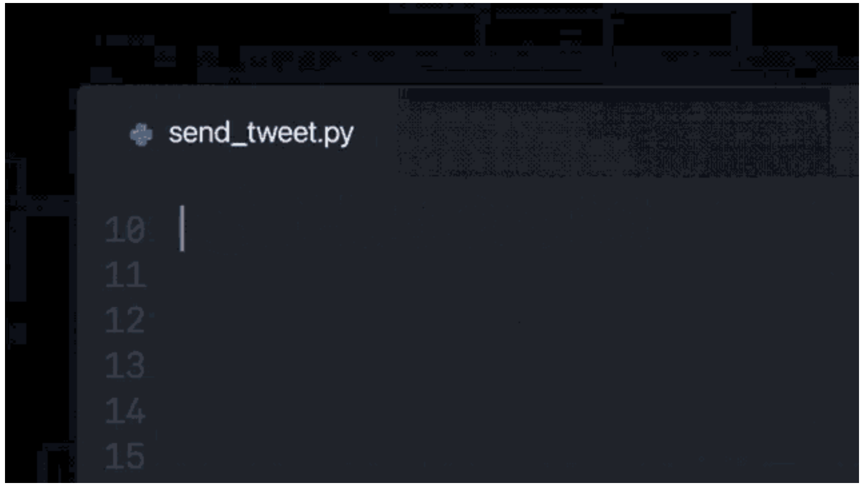

GitHub Copilot

- **LLMs are potentially powerful tools for theorem proving**

Theorem Proving as a Challenge for LLMs

- Advanced mathematical reasoning
    - Bigger models are not sufficient
    - May need formal representations in proof assistants

- Rigorous evaluation w/o hallucination
    - LLMs are hard to evaluate
    - LLMs tend to hallucinate
    - Relatively easy to check if formal proofs are correct

LLMs for Theorem Proving: Existing Work and Barriers

OpenAI

Solving (some) formal math olympiad problems

LEAN

Polu and Sutskever, GPT-f, 2020
Han et al., PACT, 2022
Polu et al., 2023
Lample et al., HTPS 2022

Meta AI

Teaching AI advanced mathematical reasoning

November 3, 2022

LLMs for Theorem Proving: Existing Work and Barriers

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Polu and Sutskever, GPT-f, 2020	✗	✗	✗	✗	774M	40K on GPU
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LeanDojo (ours)	✓	✓	✓	✓	**517M**	**120 on GPU**

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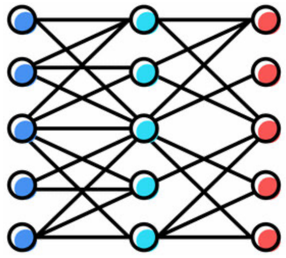
Give researchers access to state-of-the-art LLM-based provers with modest computational costs

LeanDojo

Lean (Lean 3 or Lean 4)



Machine learning model

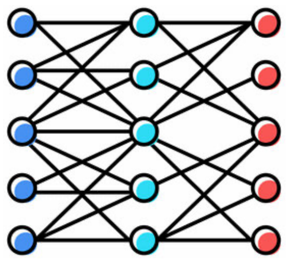
LeanDojo

Data extraction

LeanDojo Benchmark

- 98,641 theorems and proofs
- 217,639 tactics
- 129,162 premises

Lean



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LeanDojo

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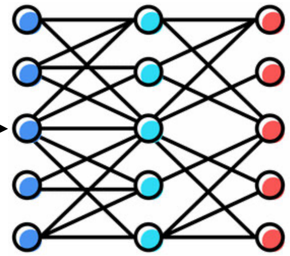

Lean

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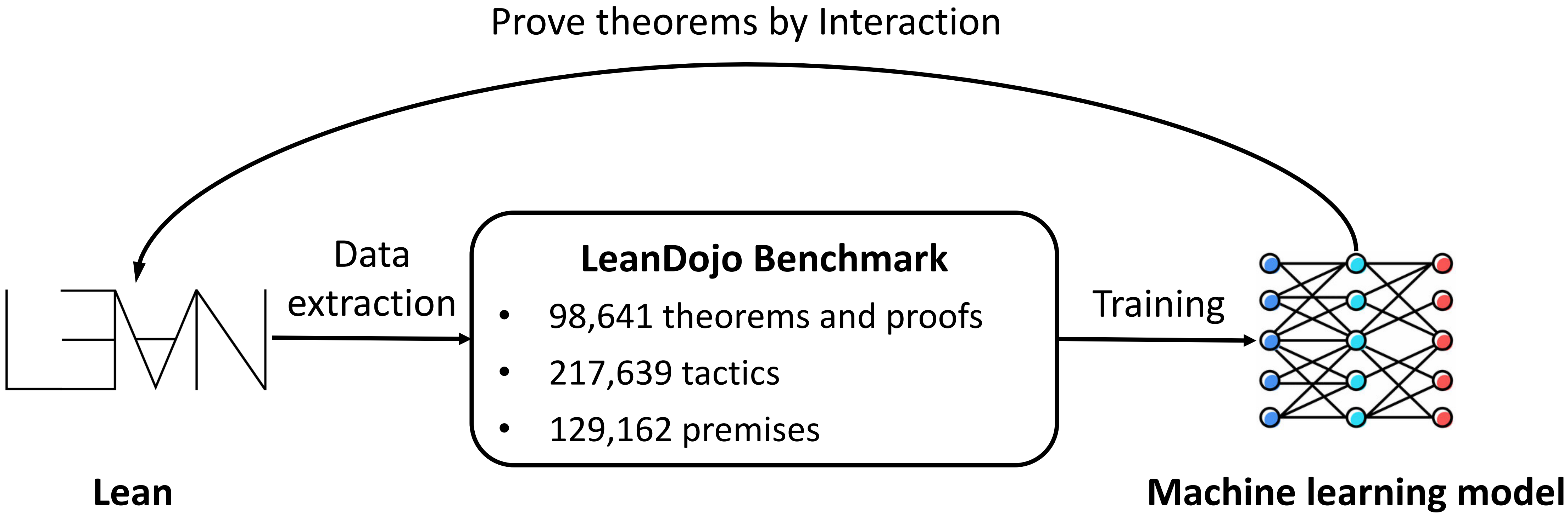
LeanDojo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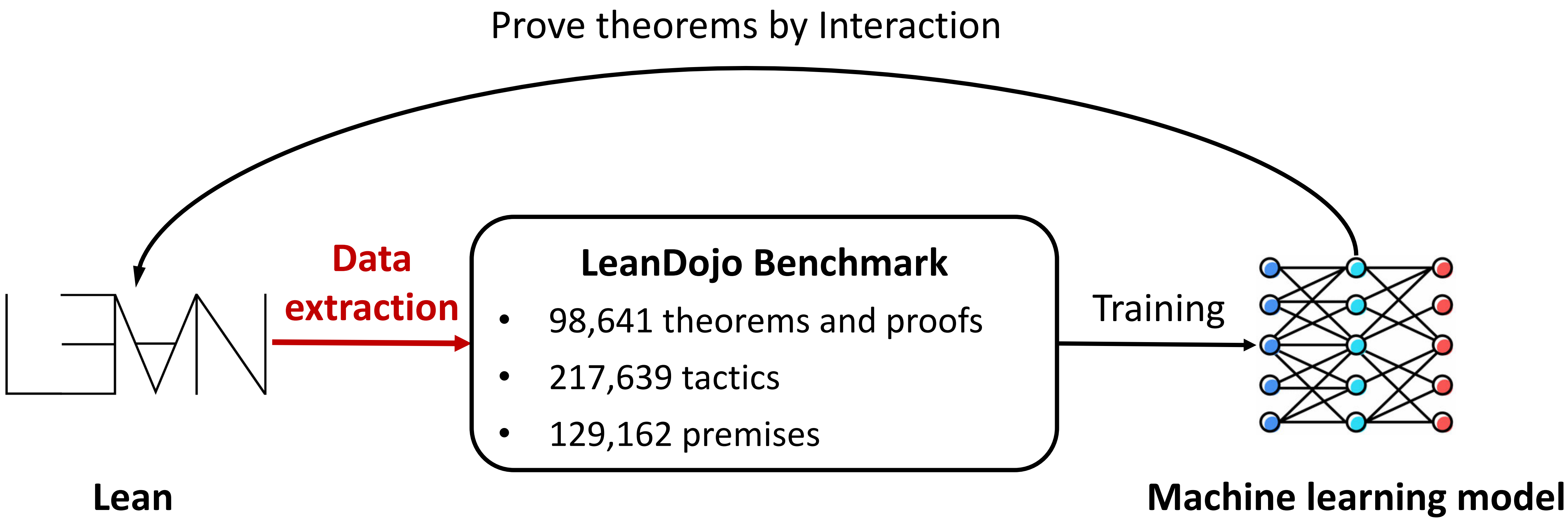

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Extracting States and Tactics

- Need **(state, tactic)** pairs for training
  - Tactics could be obtained by parsing the Lean source code into ASTs
  - Proof states are not available in the code

```
theorem gcd_self (n : nat) : gcd n n = n :=
begin
  cases n,
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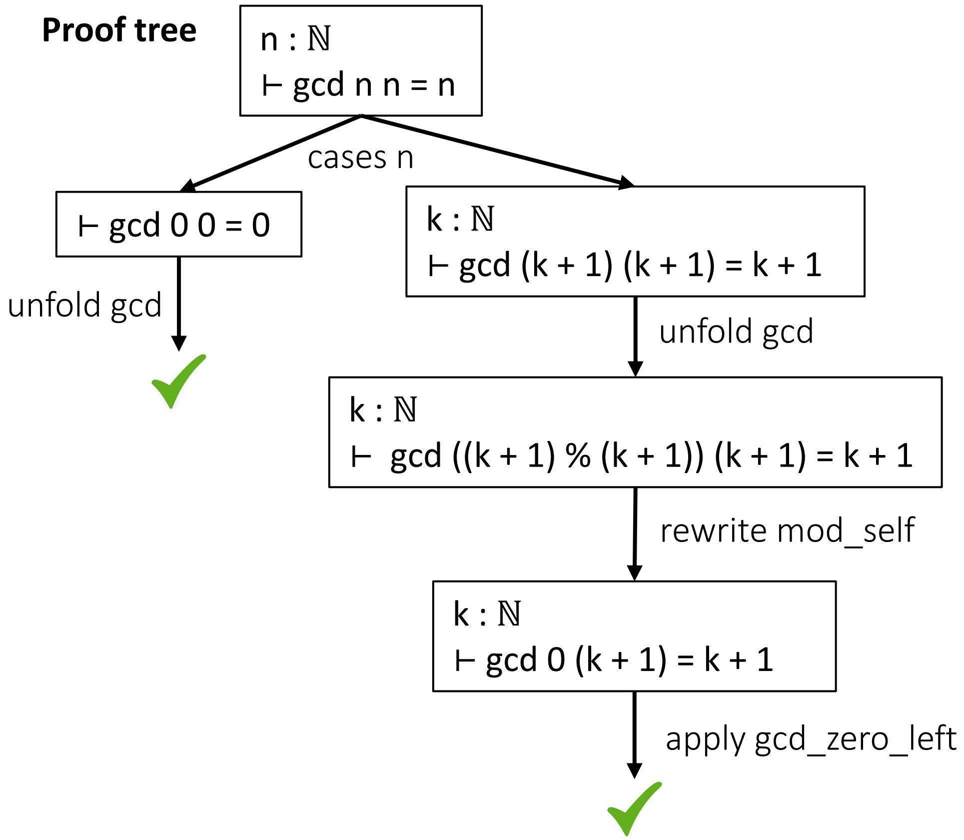

Extracting Premises in the Same File

- Tactics rely on **premises**
  - Lemmas
  - Definitions

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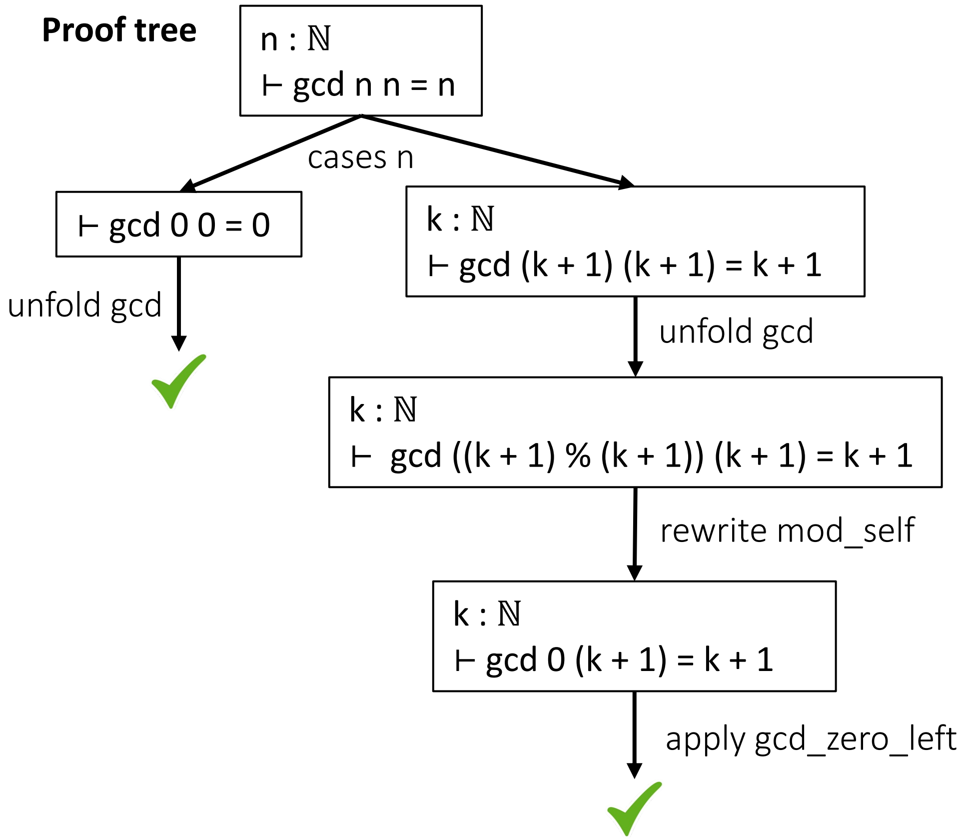

Extracting Premises in the Same File

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  - Definitions

data/nat/gcd.lean

```
def gcd : nat → nat → nat                          -- gcd z y
| 0        y := y                                  -- Case 1: z == 0
| (x + 1) y := gcd (y % (x + 1)) (x + 1)           -- Case 2: z > 0

theorem gcd_zero_left (x : nat) : gcd 0 x = x := begin simp [gcd] end

theorem gcd_self (n : nat) : gcd n n = n :=
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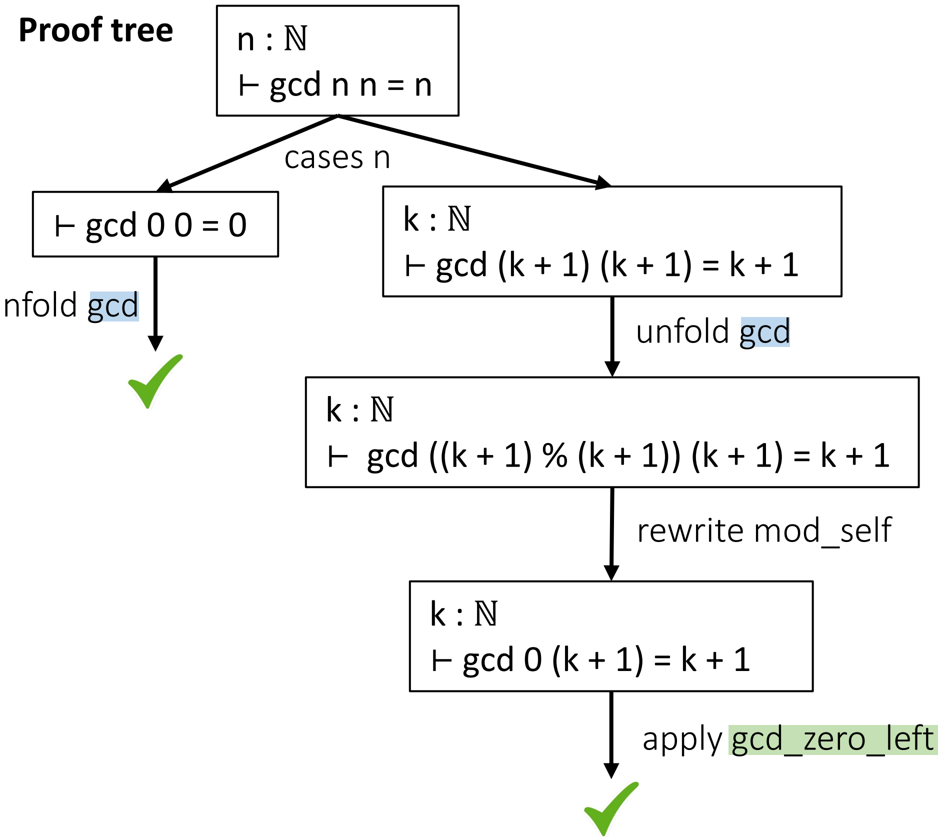

Extracting Premises from Other Files

Math library

data/nat/lemmas.lean

```
theorem mod_self (n : nat) : n % n = 0 :=
begin
  rw [mod_eq_sub_mod (le_refl _), nat.sub_self, zero_mod]
end
```

Import

data/nat/gcd.lean

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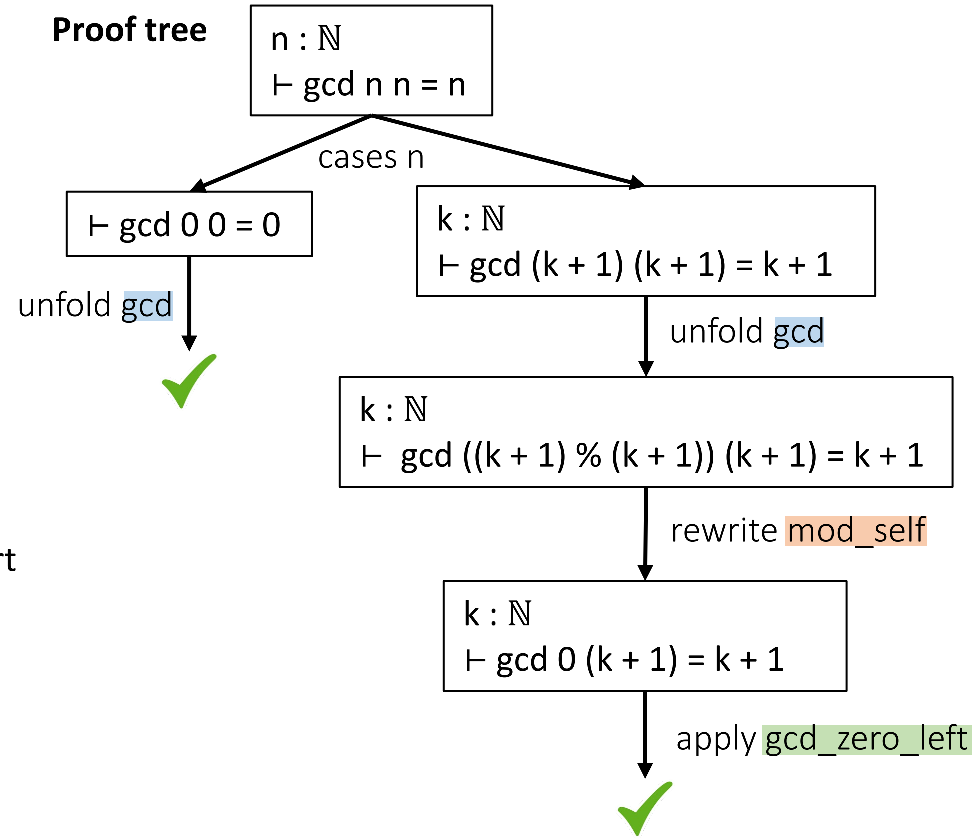
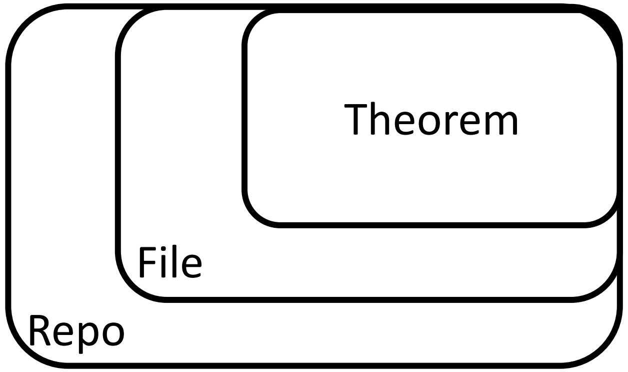
Proof tree

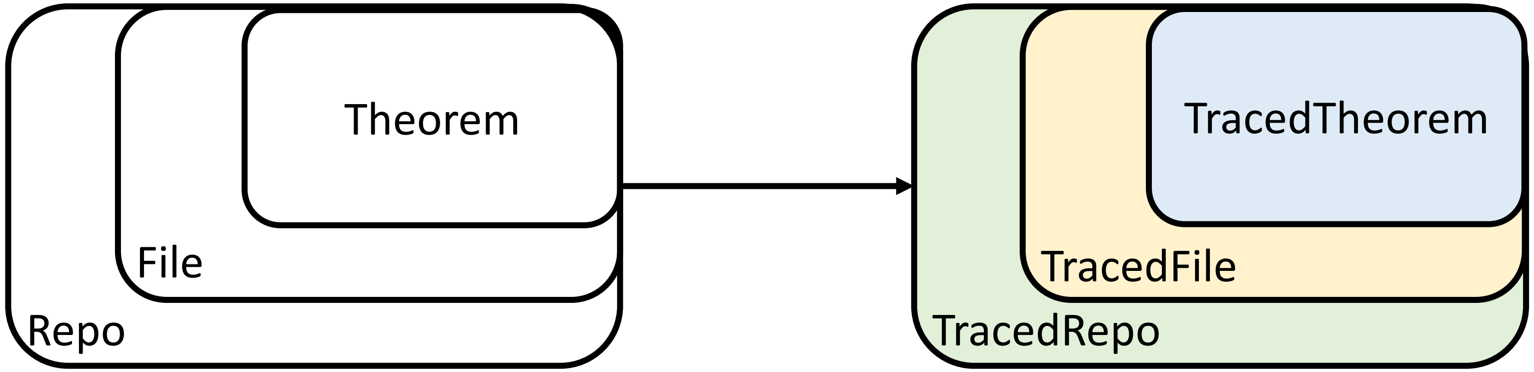


Data Extraction in LeanDojo



Arbitrary Lean repo on GitHub

Data Extraction in LeanDojo



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File dependencies, ASTs, states, tactics, premises, etc.

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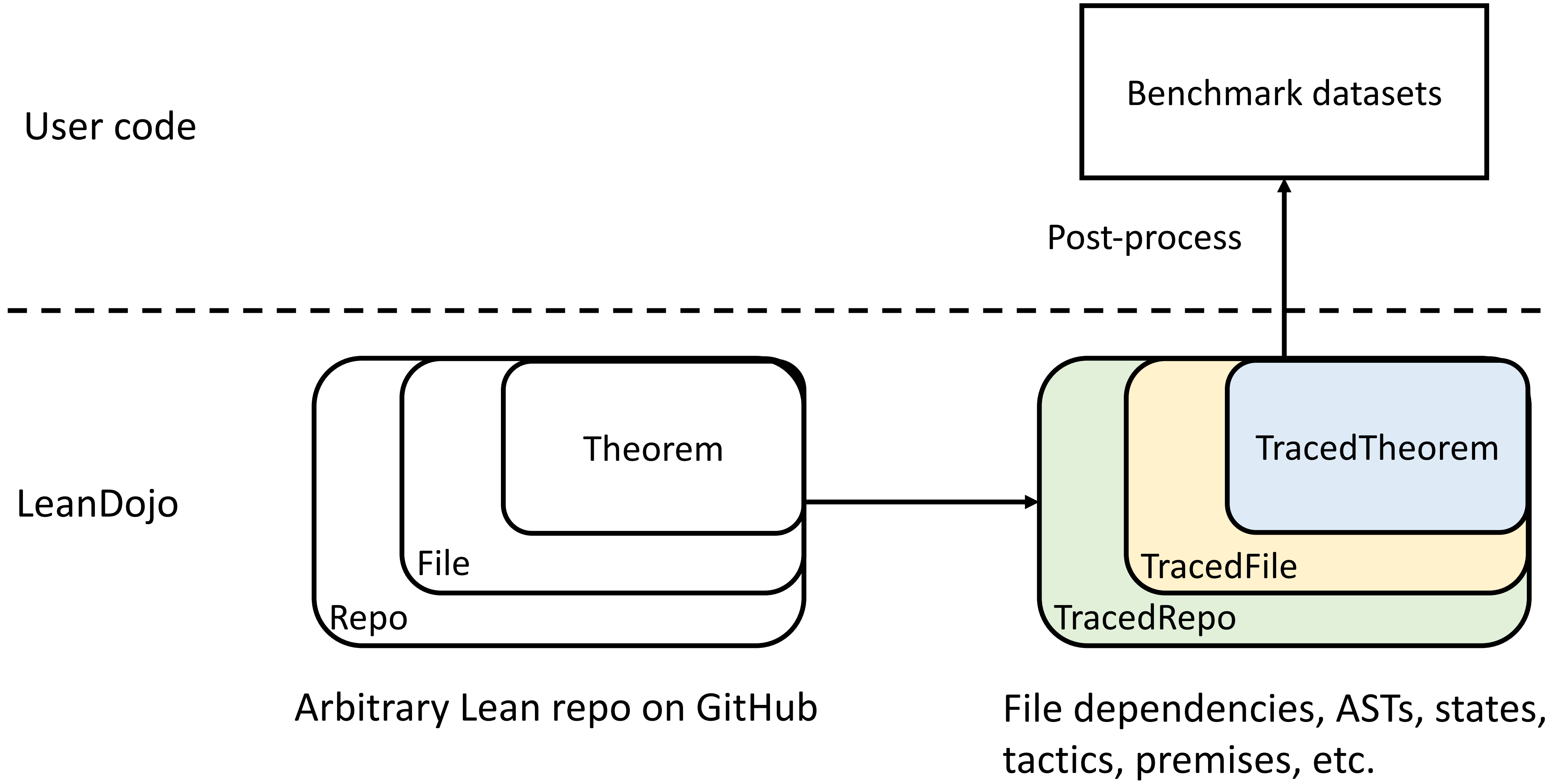
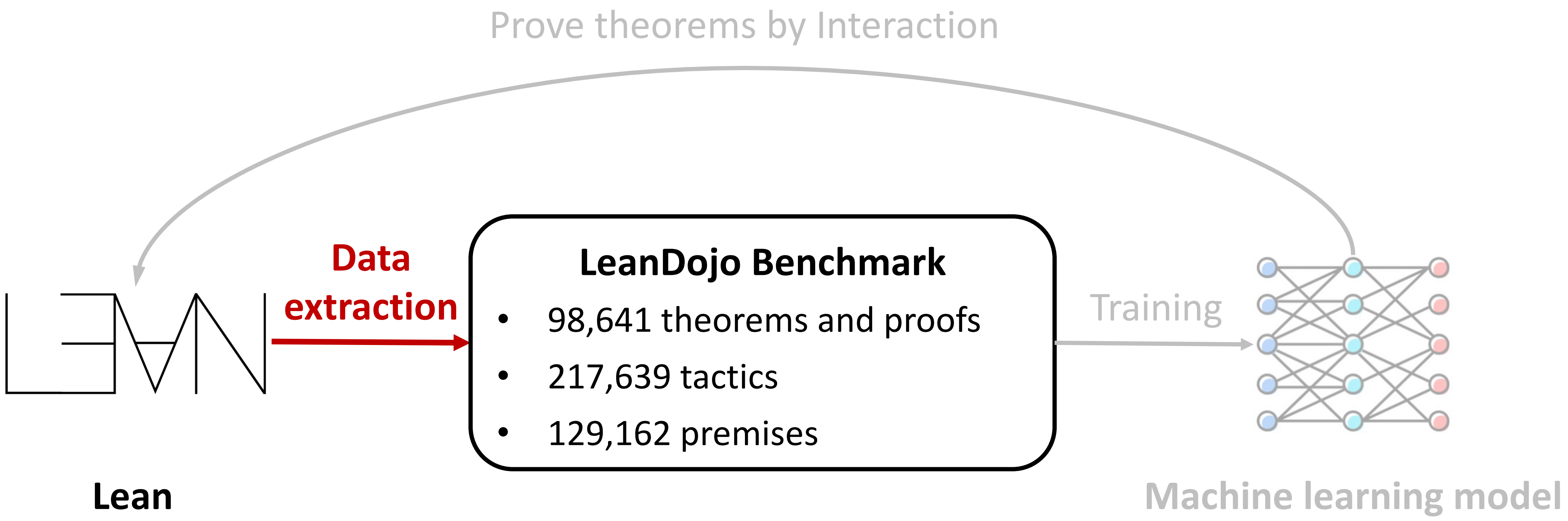


Data Extraction Demo

LeanDojo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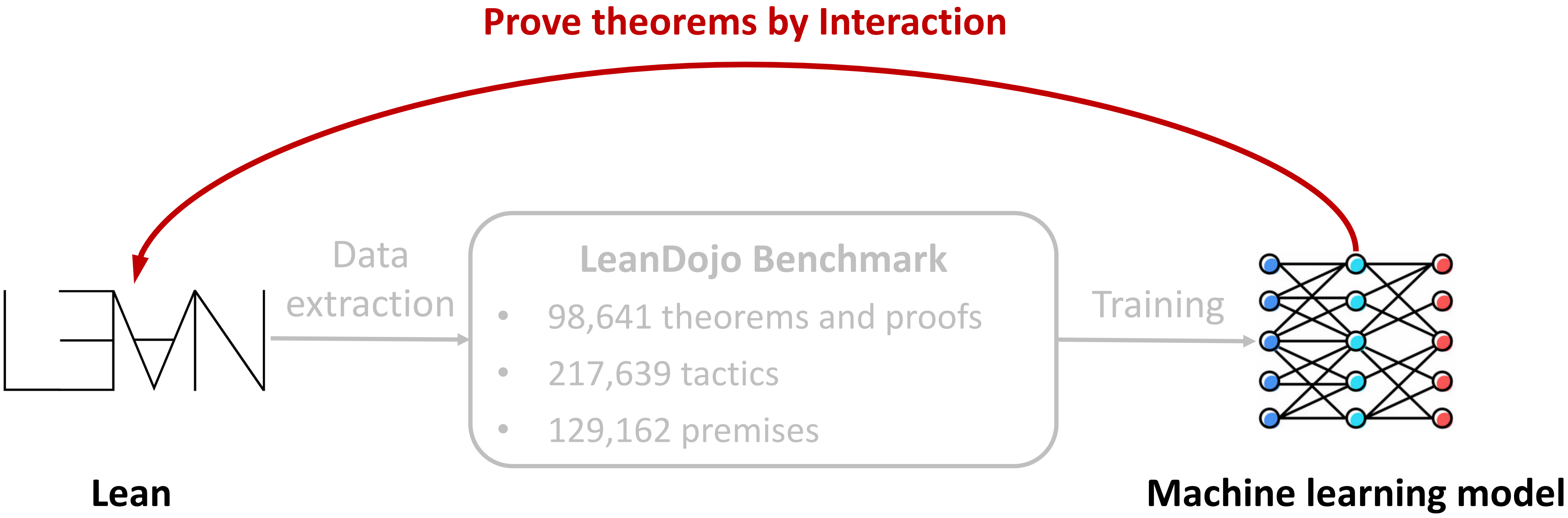
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Interacting with Lean Programmatically

- An interface for the model to observe states and run tactics
  - initialize(theorem): Given a theorem, return its initial state
  - run_tac(state, tactic): Run a tactic on a given state and return the next state

- [Demo]()

Interacting with Lean Programmatically

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- [Demo]

- **The first tool to interact with Lean 3 reliably**
  - Existing tool, lean-gym, misjudges 21% correct proofs as incorrect
  - Only 2.1% for LeanDojo

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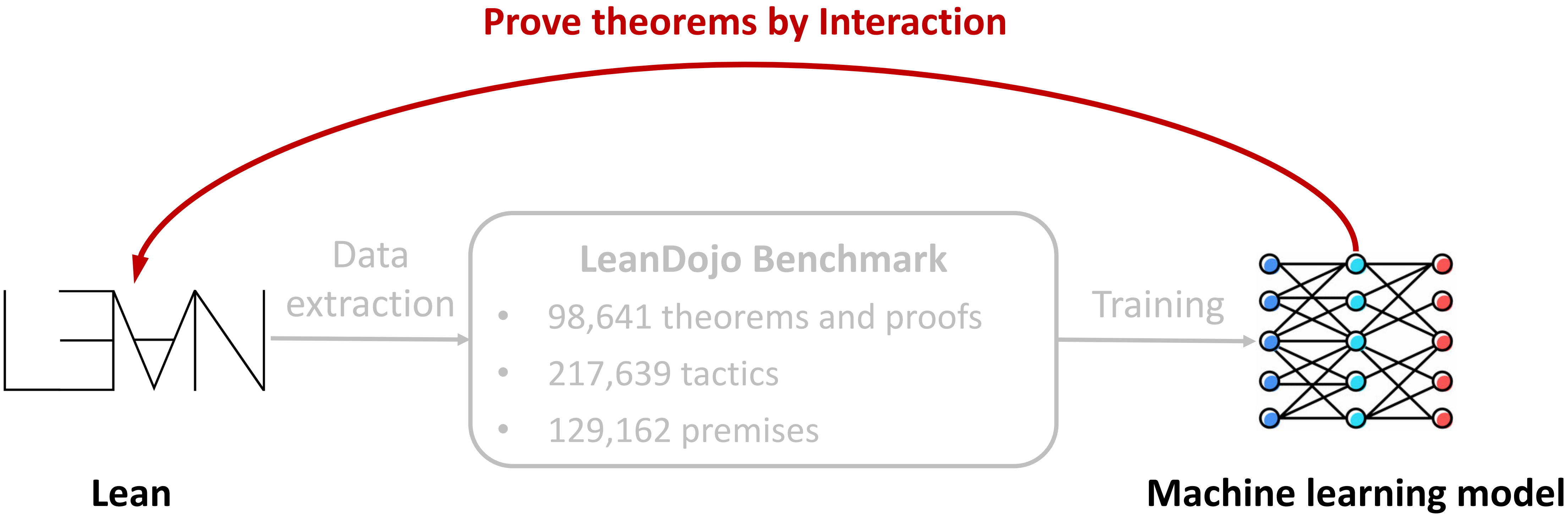
- The first tool to interact with Lean 3 reliably
  - Existing tool, lean-gym, misjudges 21% correct proofs as incorrect
  - Only 2.1% for LeanDojo
- **The first tool to interact with Lean 4**
  - Several prototypes and ongoing projects, no mature tool before LeanDojo

LeanDojo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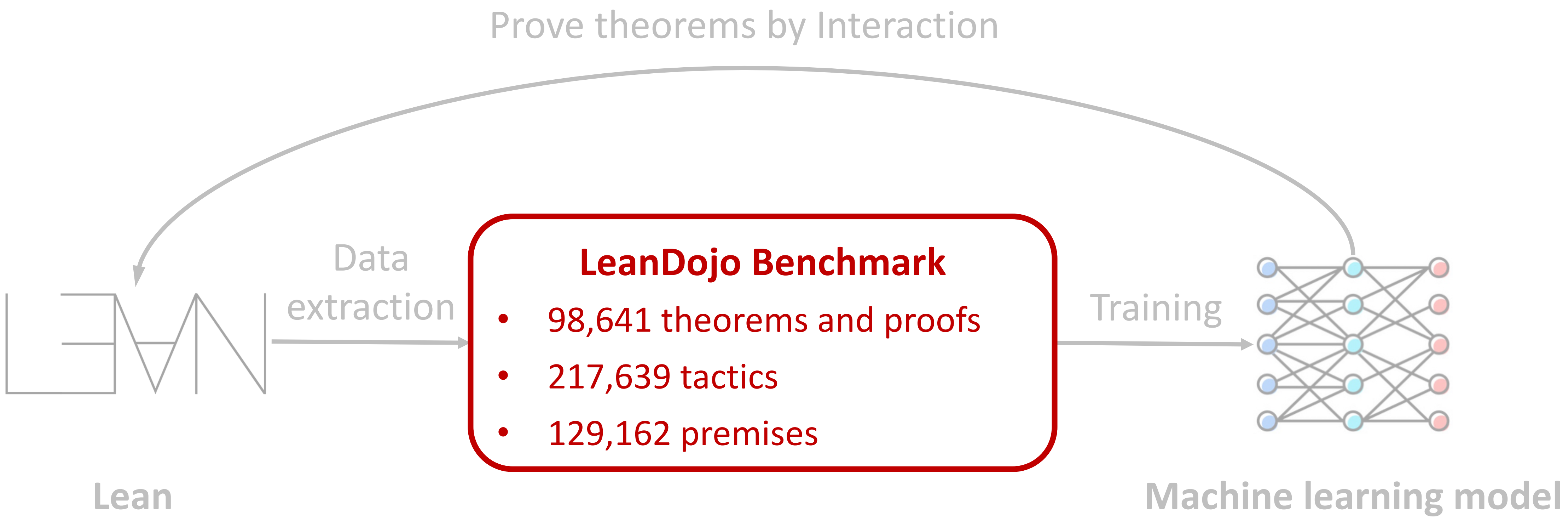
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Constructing Benchmarks using LeanDojo

- **LeanDojo Benchmark**, from mathlib on Aug 5, 2023
  - 98,641 theorems and proofs
  - 217,639 tactics
  - 129,162 premises
- **LeanDojo Benchmark 4**, from mathlib4 on Aug 10, 2023
  - 100,780 theorems and proofs
  - 209,133 tactics
  - 101,500 premises
- Easy to construct your own benchmarks

Challenging Data Split

- random: Splitting theorems into training/validation/testing randomly
- LLMs can prove seemingly nontrivial theorems by memorizing similar proofs in training

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src/algebra/quaternion.lean

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lemma conj_mul : (a * b).conj = b.conj * a.conj := begin
  ext; simp; ring_exp
end

lemma conj_conj_mul : (a.conj * b).conj = b.conj * a := begin
  rw [conj_mul, conj_conj]
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lemma conj_mul_conj : (a * b.conj).conj = b * a.conj := begin
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Test

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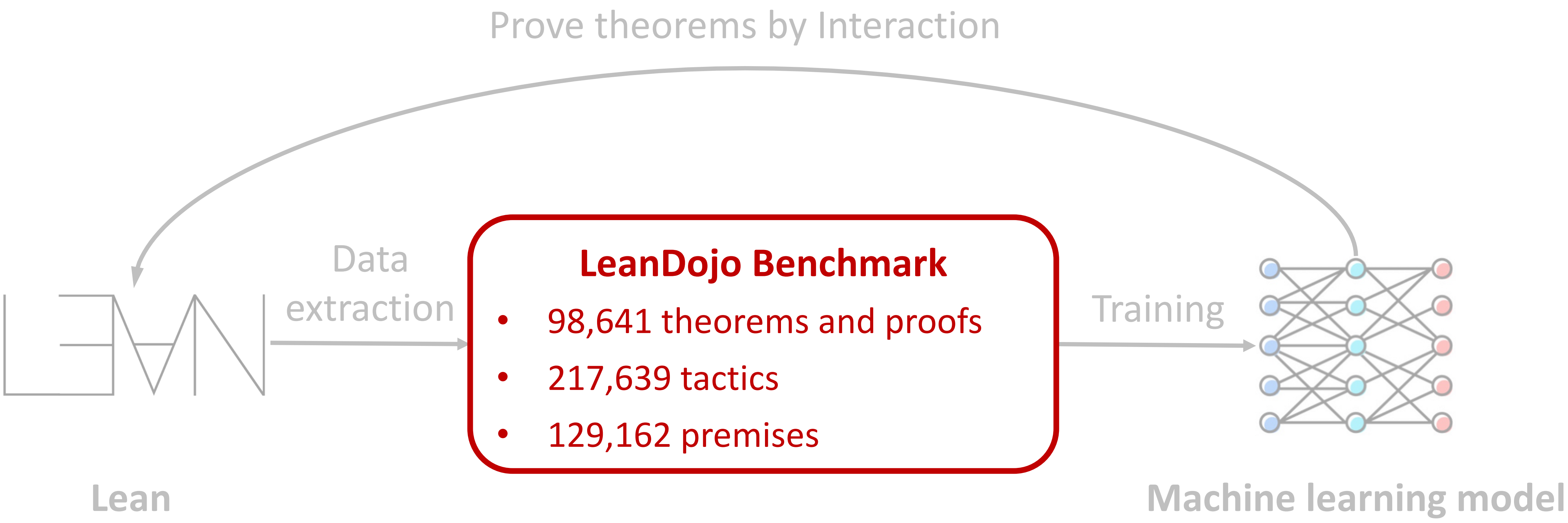
Test

```
lemma conj_mul_conj : (a * b.conj).conj = b * a.conj := begin
  rw [conj_mul, conj_conj]
end
```

- novel_premises: **Testing proofs must use >1 premise that is never used in training**

LeanDojo


Prove theorems by Interaction
Data extraction
LeanDojo Benchmark
• 98,641 theorems and proofs
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Training
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Machine learning model

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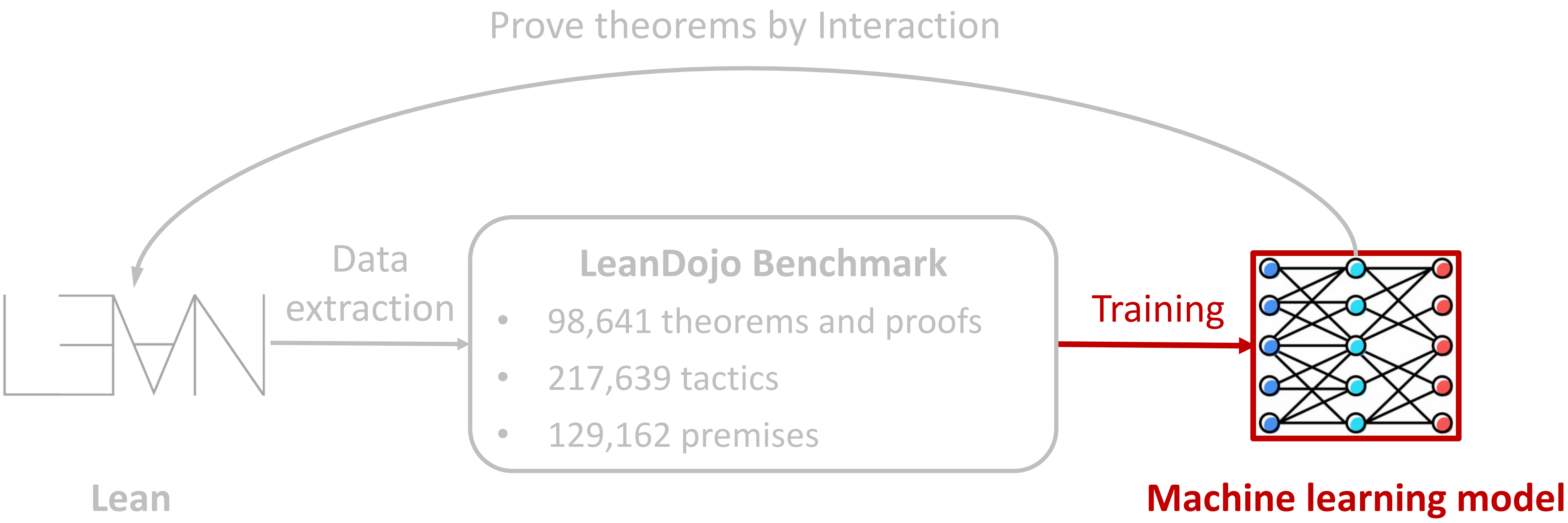
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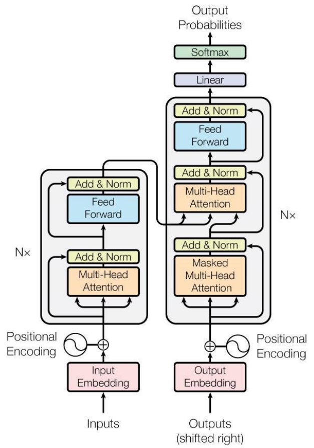
Tactic Generator in Existing LLM-Based Provers

- State -> tactic
- The model can use premises only by memorizing their names

State

k : ℕ
⊢ gcd ((k + 1) % (k + 1)) (k + 1) = k + 1



→ rewrite mod_self

Tactic

[Vaswani et al., NeurIPS 2017]

Retrieval-Augmented Prover (ReProver)

- Given a state, we retrieve premises from the set of **all accessible premises**

State

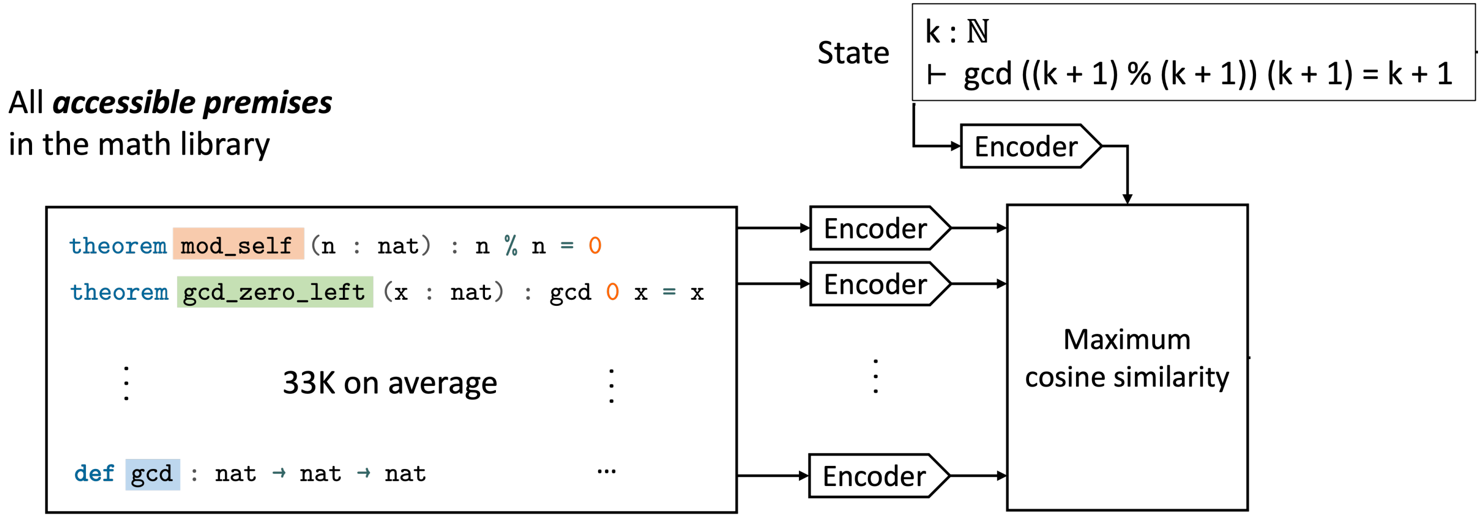
k : ℕ
⊢ gcd ((k + 1) % (k + 1)) (k + 1) = k + 1

All ***accessible premises***
in the math library

```
theorem mod_self (n : nat) : n % n = 0
theorem gcd_zero_left (x : nat) : gcd 0 x = x
   ⋮            33K on average            ⋮
def gcd : nat → nat → nat                 …
```

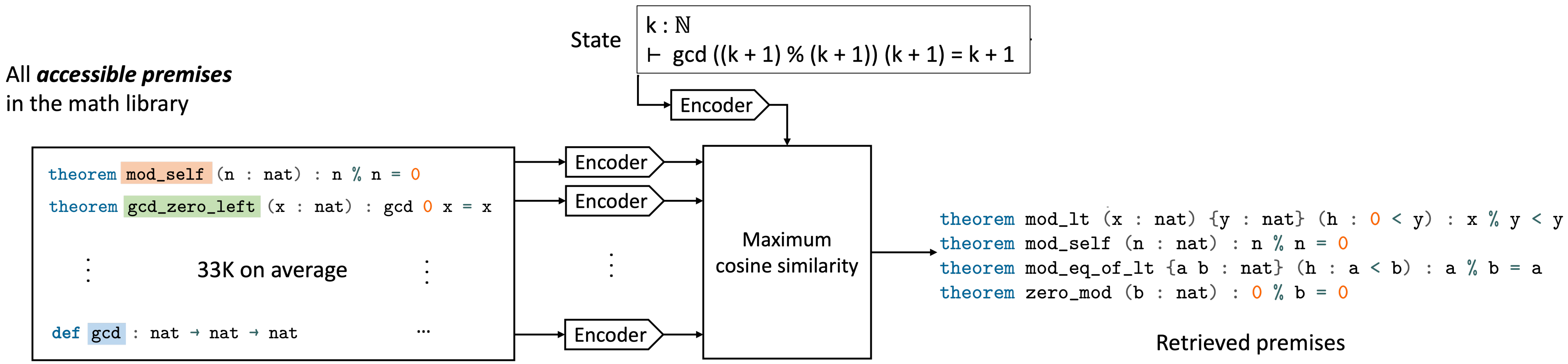
Retrieval-Augmented Prover (ReProver)

- Given a state, we retrieve premises from the set of **all accessible premises**



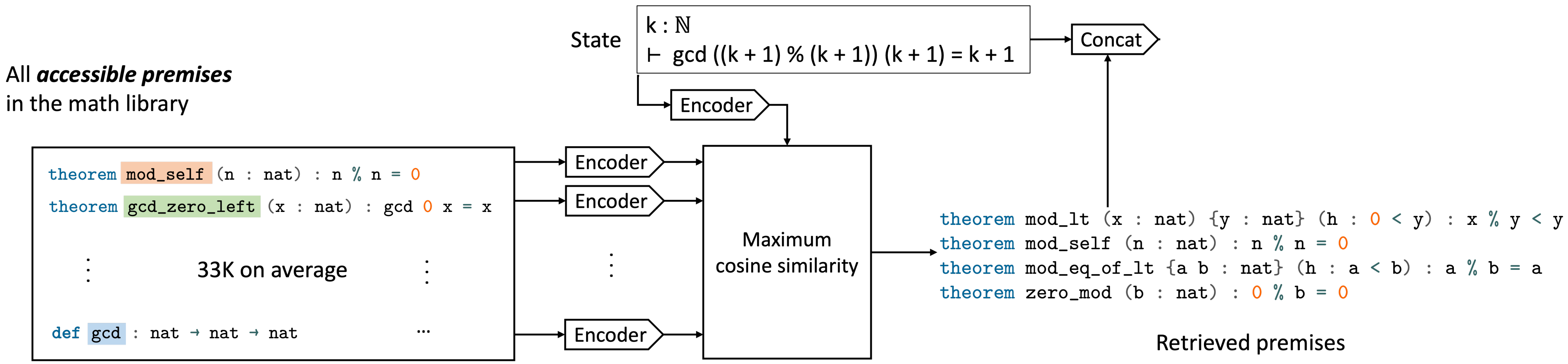
Retrieval-Augmented Prover (ReProver)

- Given a state, we retrieve premises from the set of **all accessible premises**



Retrieval-Augmented Prover (ReProver)

- Given a state, we retrieve premises from the set of **all accessible premises**
- Retrieved premises are concatenated with the state and used for tactic generation



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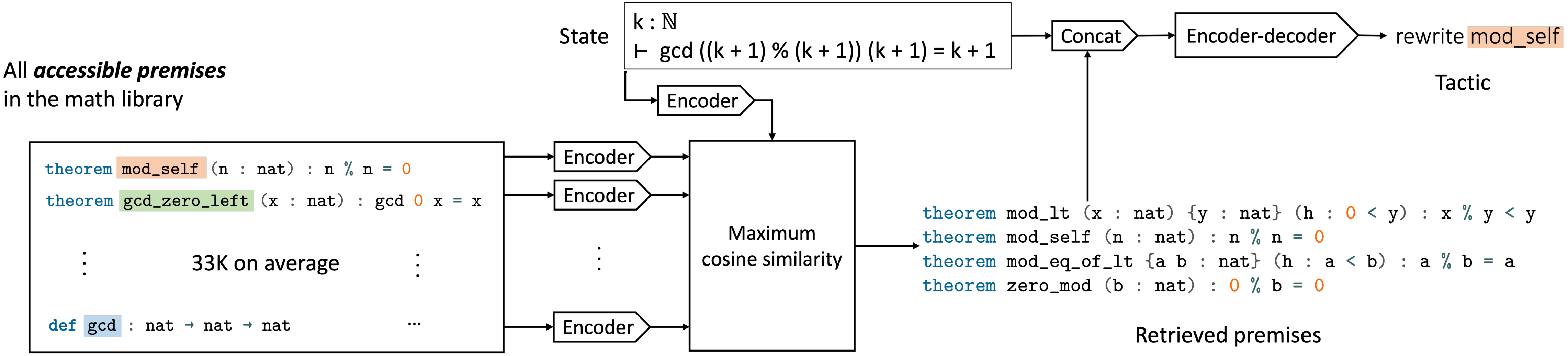
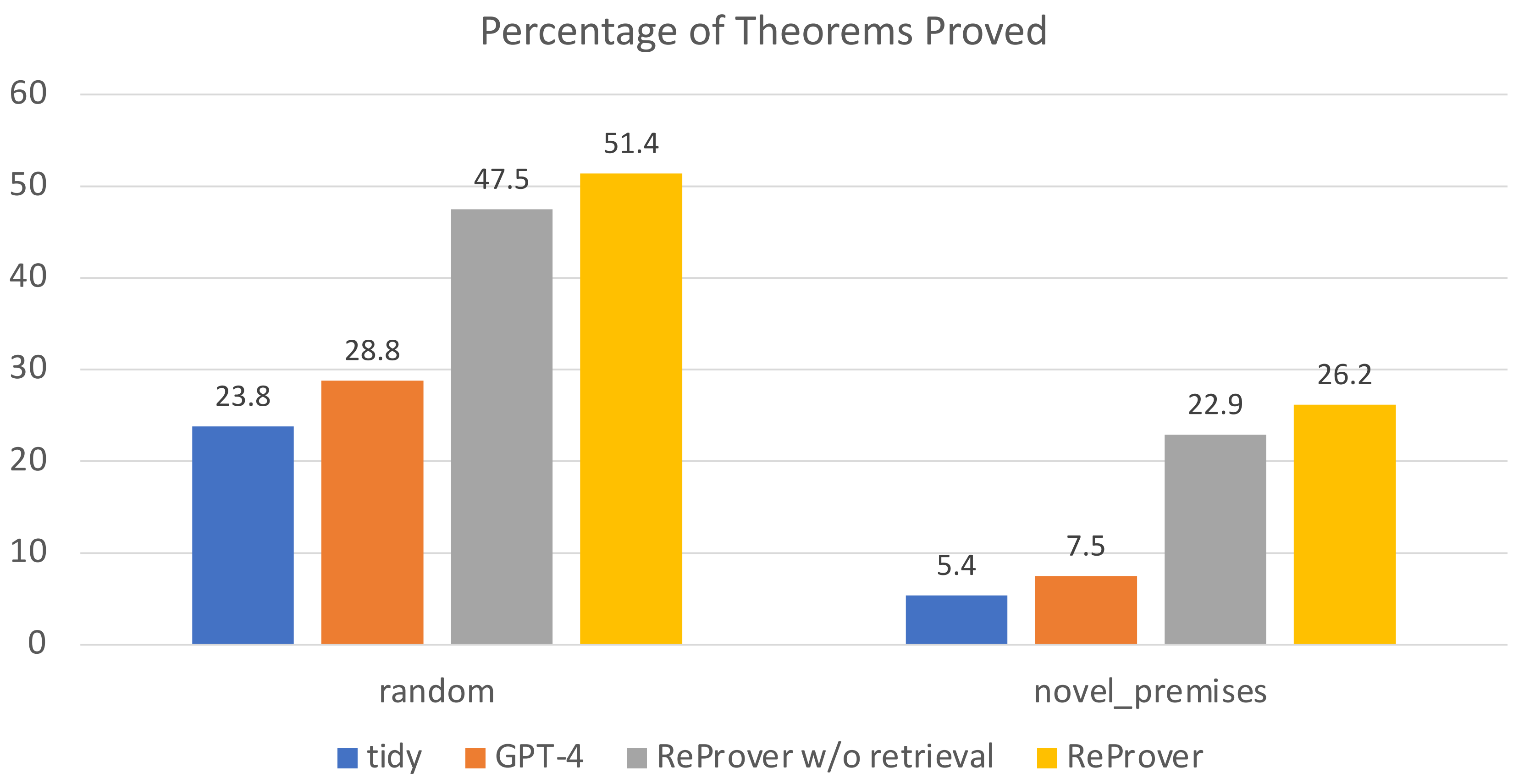
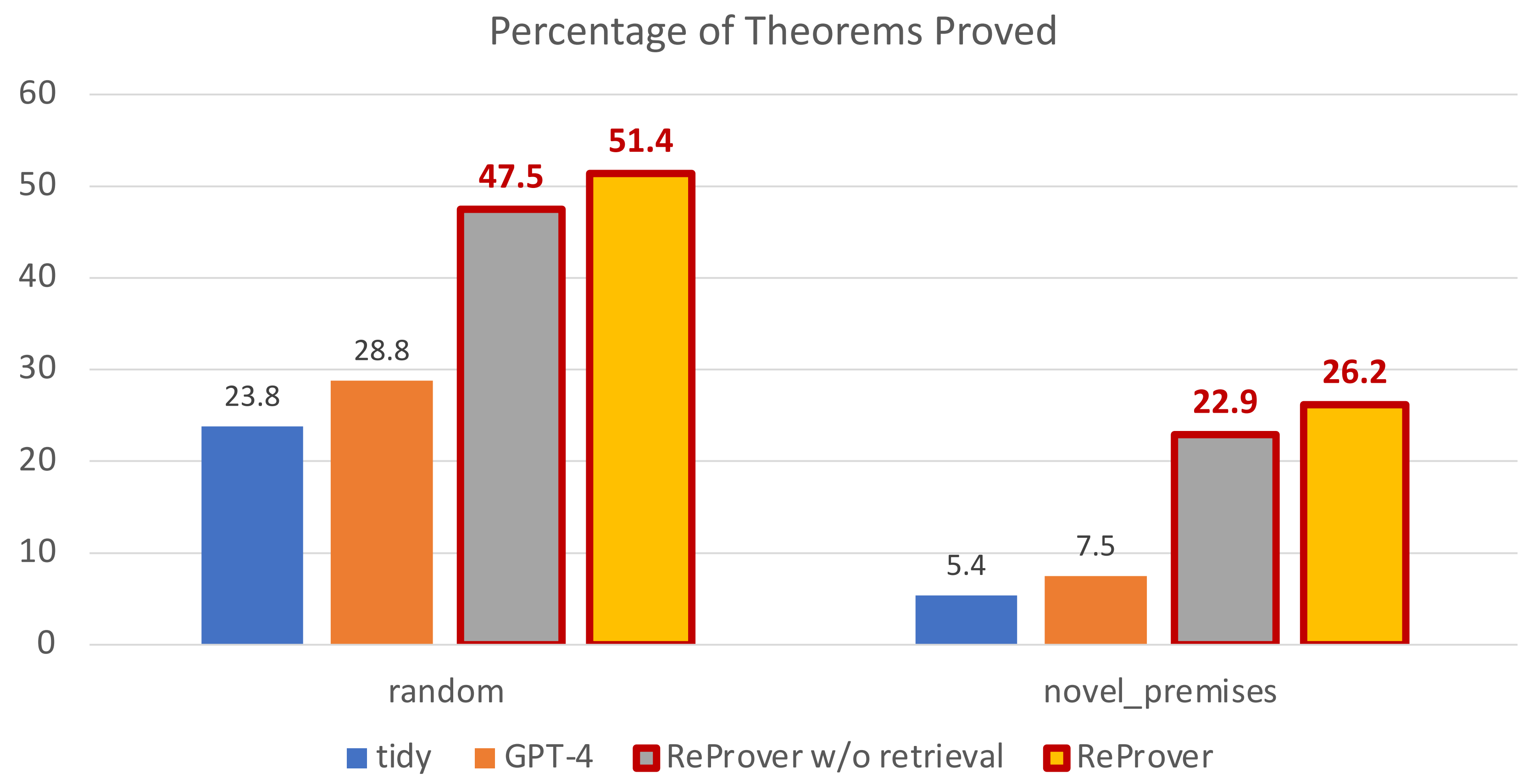


Premise Retrieval Improves Theorem Proving



Premise Retrieval Improves Theorem Proving



Discovering New Proofs on MiniF2F and ProofNet

- We evaluate the model on MiniF2F and ProofNet (in zero shot) to discover new Lean proofs

```
theorem exercise_2_3_2 {G : Type*} [group G] (a b : G) :
   g : G, b * a = g * a * b * g ¹ :=
begin
  exact  b, by simp ,
end

theorem exercise_11_2_13 (a b :  ) :
  (of_int a : gaussian_int)   of_int b → a   b :=
begin
  contrapose,
  simp,
end

theorem exercise_1_1_17 {G : Type*} [group G] {x : G} {n :  }
  (hxn: order_of x = n) :
  x ¹ = x ^ (n - 1 :  ) :=
begin
  rw zpow_sub_one,
  simp,
  rw [← hxn, pow_order_of_eq_one],
end
```

```
theorem exercise_3_1_22b {G : Type*} [group G] (I : Type*)
  (H : I → subgroup G) (hH :   i : I, subgroup.normal (H i)) :
  subgroup.normal (  (i : I), H i):=
begin
  rw infi,
  rw ←set.image_univ,
  rw Inf_image,
  simp [hH],
  haveI :=   i, (H i).normal,
  split,
  intros x hx g,
  rw subgroup.mem_infi at hx  ,
  intro i,
  apply (hH i).conj_mem _ (hx i),
end

theorem exercise_3_4_5a {G : Type*} [group G]
  (H : subgroup G) [is_solvable G] : is_solvable H :=
begin
  apply_instance,
end
```

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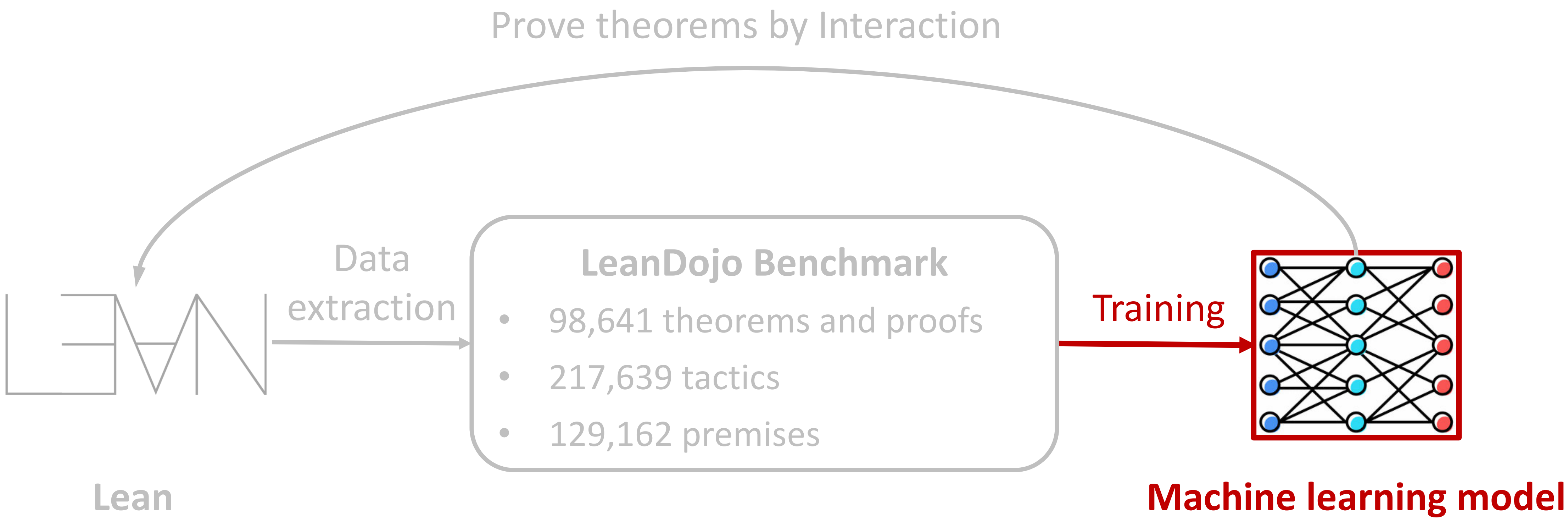
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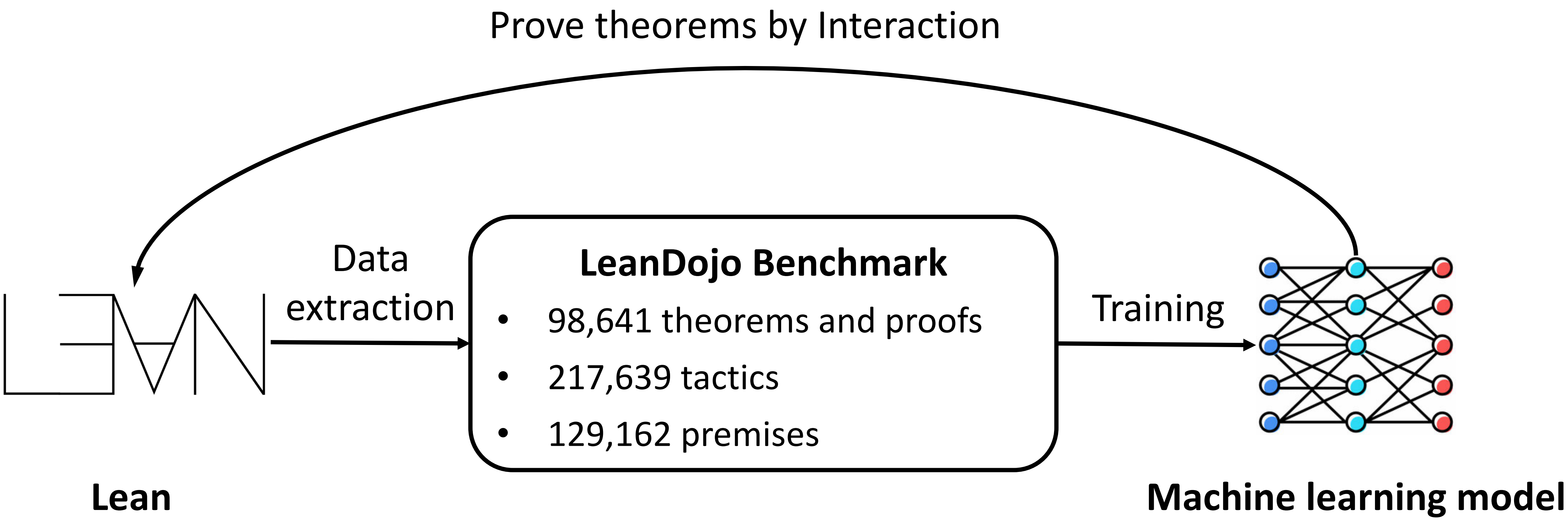
- https://leandojo.org
- Tutorial @ NeurIPS 2023

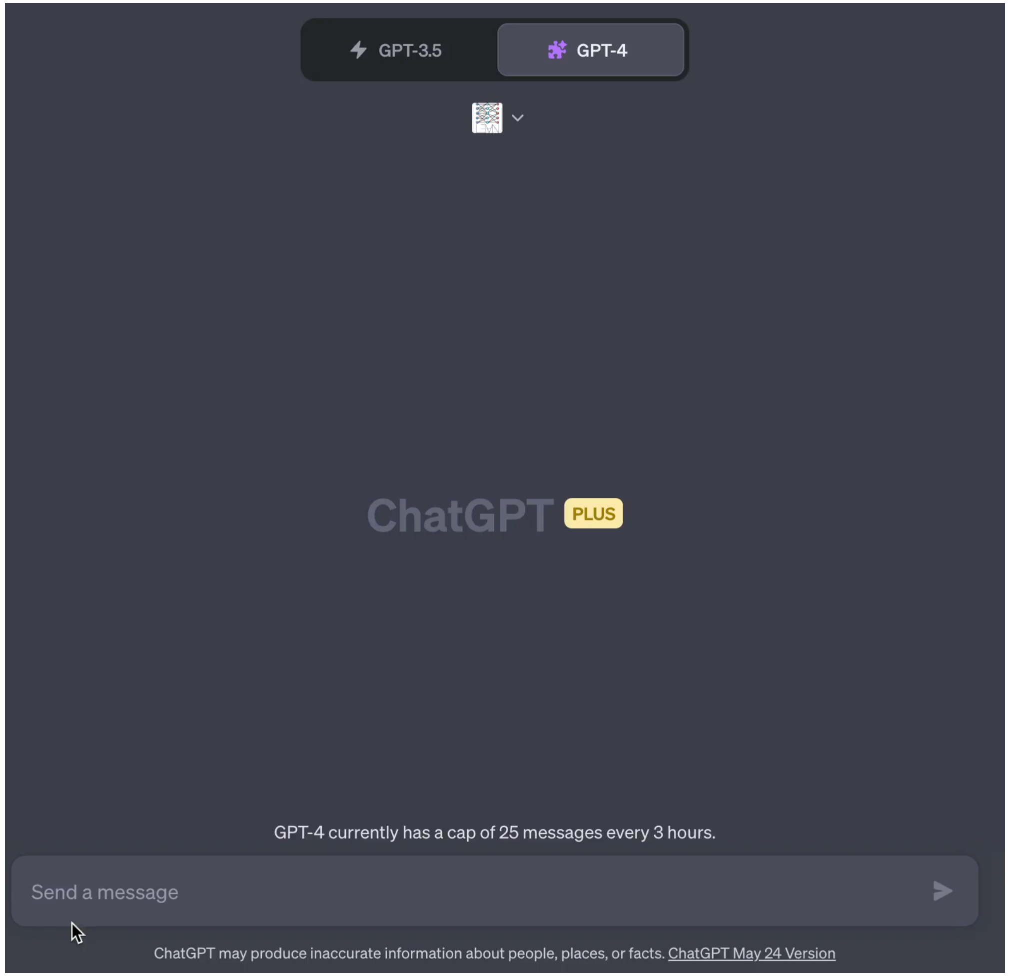


Future Direction: GPT for Theorem Proving in Lean

- ChatGPT can use LeanDojo to interact with Lean

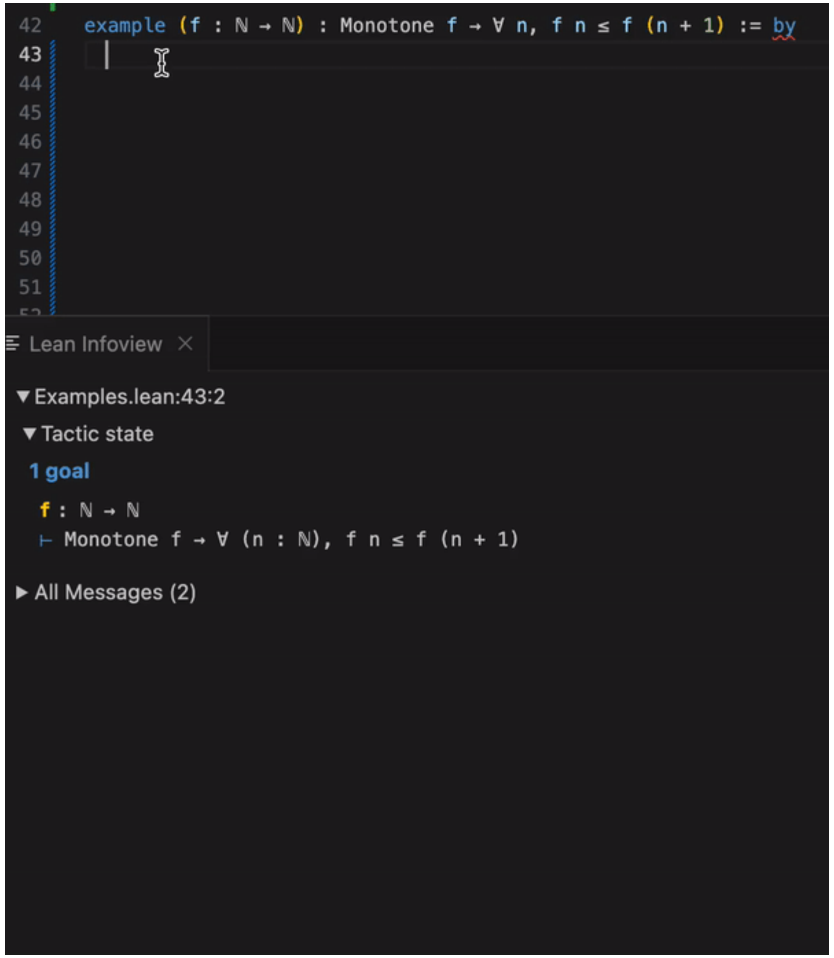
- Strengths:
  - Interleave informal math with formal proofs
  - Explain and correct errors
  - Steerable via prompt engineering

- Limitations:
  - Hallucinating informal math
  - Unable to prove nontrivial theorems



Future Direction: Tools for Lean Users

- We focus on enabling machine learning researchers to work on theorem proving
- LeanDojo is also useful for building practical proof automation tools for Lean users
- LLMStep
    - Work by Sean Welleck and Rahul Saha
    - Finetune LLMs for tactic generation on LeanDojo Benchmark
    - Integrate into Lean's VSCode workflow



Team

Aidan Swope

Alex Gu

Rahul Chalamala

Peiyang Song

Shixing Yu

Saad Godil

Ryan Prenger

Anima Anandkumar

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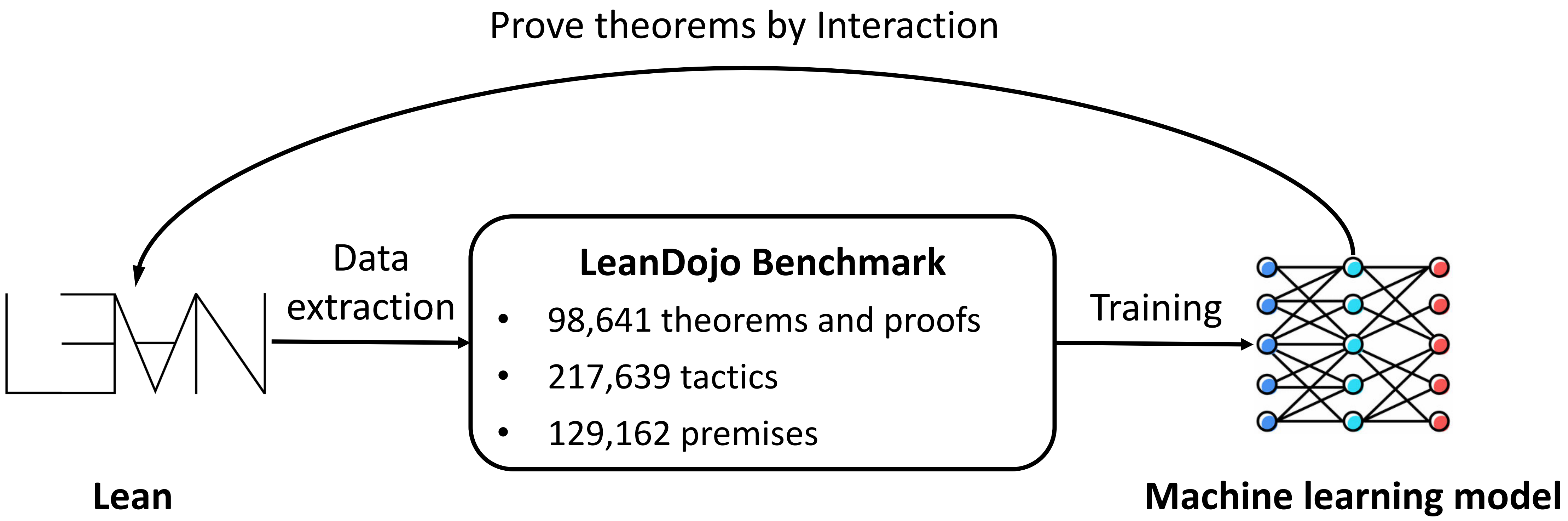
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