Studies of Mathematical Practice and Lessons for ATP

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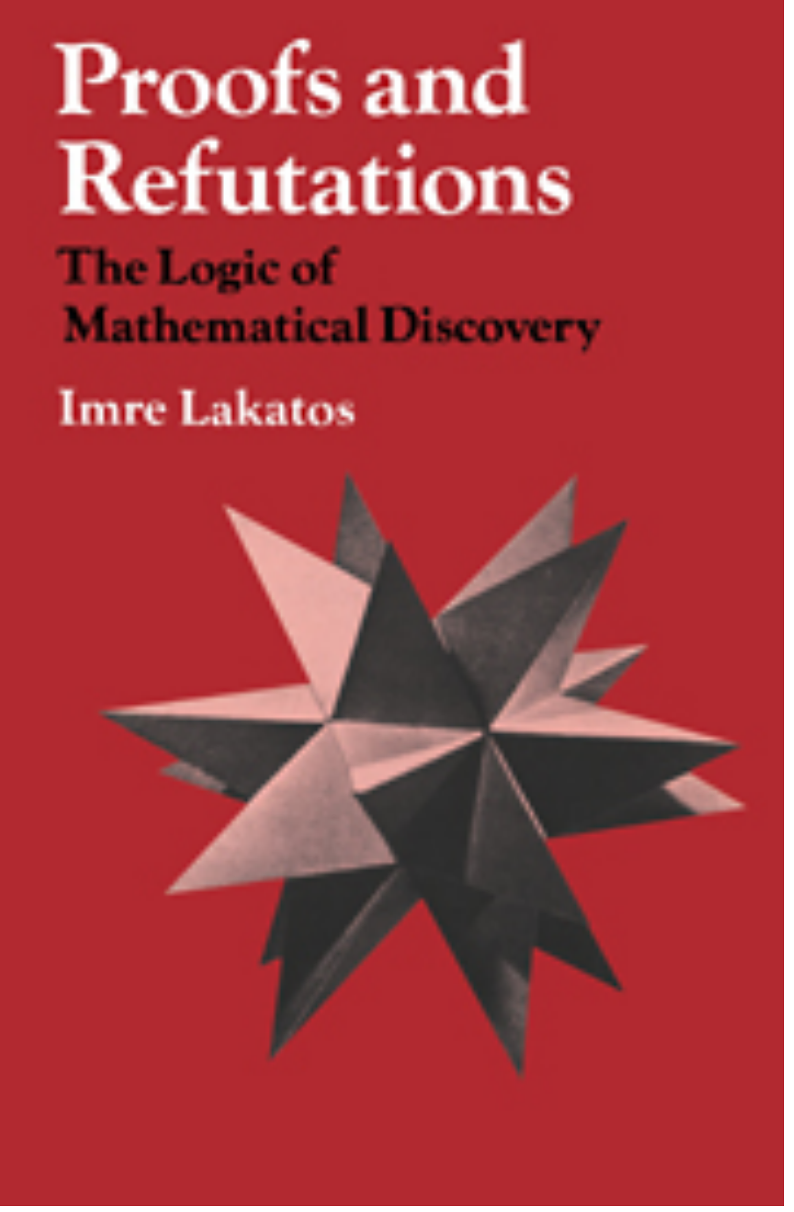
In collaboration with Ursula Martin, University of Oxford



APPENDIX 2

THE DEDUCTIVIST VERSUS THE HEURISTIC APPROACH

1. *The Deductivist Approach*

Euclidean methodology has developed a certain obligatory style of presentation. I shall refer to this as 'deductivist style'. This style starts with a painstakingly stated list of *axioms*, *lemmas* and/or *definitions*. The axioms and definitions frequently look artificial and mystifyingly complicated. One is never told how these complications arose. The list of axioms and definitions is followed by the carefully worded *theorems*. These are loaded with heavy-going conditions; it seems impossible that anyone should ever have guessed them. The theorem is followed by the *proof*.

The student of mathematics is obliged, according to the Euclidean ritual, to attend this conjuring act without asking questions either about the background or about how this sleight-of-hand is performed. If the student by chance discovers that some of the unseemly definitions are proof-generated, if he simply wonders how these definitions, lemmas and the theorem can possibly precede the proof, the conjuror will ostracize him for this display of mathematical immaturity.[1]

In deductivist style, all propositions are true and all inferences valid. Mathematics is presented as an ever-increasing set of eternal, immutable truths. Counterexamples, refutations, criticism cannot possibly enter. An authoritarian air is secured for the subject by beginning with disguised monster-barring and proof-generated definitions and with the fully-fledged theorem, and by suppressing the primitive conjecture, the refutations, and the criticism of the proof. Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem in the course of the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility.[2]

Deductivist Approach:

- Hides the struggle, hides the adventure
- Disconnect between concepts, entities, axioms, theorems and proofs; i.e., discovery and justification
- Has an authoritarian air
- Focus on soundness rather than understandability

Automated Theorem Proving:

- Hides the struggle, hides the adventure
- Disconnect between concepts, entities, axioms, theorems and proofs; i.e., discovery and justification
- Has an authoritarian air
- Focus on soundness rather than understandability

At an event in 2012 organised by Martin and Pease, leading mathematicians flagged the importance of collaborative systems that "think like a mathematician", handle unstructured approaches such as the use of "sloppy" natural language and the exchange of informal knowledge and intuition not recorded in papers, and engage diverse researchers in creative problem solving. This accords with work of cognitive scientists, sociologists, philosophers and the narrative accounts of mathematicians themselves, which highlight the paradoxical nature of mathematical practice — while the goal of mathematics is to discover mathematical truths justified by rigorous argument, mathematical discovery involves "soft" aspects such as creativity, informal argument, example, error and analogy.

events.inf.ed.ac.uk/sicsa-mcp/

“if we wish to teach computers to find proofs, it is likely to be a good idea to reflect on how we do so ourselves.”

W. T. Gowers. *Rough structure and classification.* GAFA (Geometric And Functional Analysis), Special volume – GAFA2000(1–0), 2000.

How do humans do mathematics?

- What are the patterns of communication?
- What do they talk about?
- How do they explain things?
- What do they value?
- How do they use examples?

Research Question	How do mathematicians communicate?	What do mathematicians talk about?	How do mathematicians explain things?	What do mathematicians value?	How do mathematicians use examples?
Data	Euler's characteristic (1750) and Cauchy's proof (1811)			JOURNAL OF ALGEBRA; mathoverflow	MATHEMATICS
Methodology	rational reconstruction	grounded theory (data-driven)	analysis of keywords and random sampling (theory-driven)	keyword search	grounded theory; user testing; machine learning (triangulation)
Results	A set of heuristics	A typology of comments, with hierarchy and timestamps	A set of hypotheses with support or attacks	Comparison between values in the front and back stage	A theory of example-use
Computational System	HRL; TM; Lakatos Games	—	—	—	EgBot

Features of our approach

- Empirical data - usually online fora
- “Everyday” mathematics
- Collaborative mathematics
- “Backstage” mathematics (cf Hersh)

- A range of methodologies — including data-driven, theory-driven, user-driven, using a computational lens to test and extend

The nature of mathematical collaboration is changing

- A number of senior mathematicians produce influential and widely read blogs.
- Discussion fora allow rapid informal interaction and problem-solving
- Online forums and blogs for informal mathematical discussion reveal some of the ‘back’ of mathematics:

‘mathematics as it appears among working mathematicians, in informal settings, told to one another in an office behind closed doors’

Hersh, R. (1991). Mathematics has a front and a back. Synthese, 88:127–133.

‘it has provided, for possibly the first time ever (though I may well be wrong about this), the first fully documented account of how a serious research problem was solved, complete with false starts, dead ends etc.’

Gowers, T. (2009). Polymath1 and open collaborative mathematics. http://gowers.wordpress.com/2009/03/10/.

Frontstage mathematics

Corollary 3.3. *Let ν and $\tilde{\nu}$ be the equal-slices and non-degenerate equal-slices measures on $[k]^n$, respectively. Then for any set $A \subset [k]^n$ we have $|\nu(A) - \tilde{\nu}(A)| \leq k^2/n$*

Proof. It follows from Lemma 3.2 that the probability that a slice is degenerate is at most k^2/n. Therefore, if A is a set that consists only of non-degenerate sequences, then its non-degenerate equal-slices measure is $(1-c)^{-1}$ times its equal-slices measure, for some $c < k^2/n$. Therefore, for such a set, $0 \leq \tilde{\nu}(A) - \nu(A) = c\tilde{\nu}(A) \leq k^2/n$. If A consists only of degenerate sequences, then $0 \leq \nu(A) - \tilde{\nu}(A) = \nu(A) \leq k^2/n$. The result follows, since if one takes a union of sets of the two different kinds, then the differences cancel out rather than reinforcing each other. □

For later use, we slightly generalize Lemma 3.2.

Lemma 3.4. *Let x be chosen randomly from $[k]^n$ using the equal-slices distribution. Then the probability that fewer than m coordinates of x are equal to k is at most mk/n.*

Proof. Let P be as in the proof of Lemma 3.2. This time we are interested in the probability that $p_{k-1} \geq n+k-m$. The number with $p_{k-1} = n+k-s$ is $\binom{n+k-s-1}{k-2}$, which is at most $\binom{n+k-2}{k-2}$, which as we noted in the proof of Lemma 3.2 is at most $\frac{k}{n}\binom{n+k-1}{k-1}$. The result follows. □

Corollary 3.5. *Let x be chosen randomly from $[k]^n$ using the equal-slices distribution. Then the probability that there exists $j \in [k]$ such that fewer than m coordinates of x are equal to j is at most mk^2/n.*

Proof. This follows immediately from Lemma 3.4. □

Backstage mathematics

31. Gil, a quick remark about Fourier expansions and the $k = 3$ case. I want to explain why I got stuck several years ago when I was trying to develop some kind of Fourier approach. Maybe with your deep knowledge of this kind of thing you can get me unstuck again.

1. A quick question. Furstenberg and Katznelson used the Carlson–Simpson theorem in their proof. Does anyone know that proof well enough to know whether the Carlson–Simpson theorem might play a role here? If so, I could add

This looks weird enough that it's probably a wrong idea, but I still feel there may be a "natural" probabilistic way to do it.

Here's an attempt to throw the spanner in the works of #32.

So it looks to me as though it would be disastrous to take the uniform distribution over lines with some fixed number of wildcards (unless, perhaps, one had done some more preprocessing to get a stronger property than mere richness).

27. I was rather pleased with the "conjecture" in the final paragraph of comment 22, but have just noticed that it is completely false, at least if you interpret it in the most obvious way. Indeed, if you take both $\mathcal{A}$ and $\mathcal{B}$ to consist of *all*

“Is this data representative of mathematical practice?”

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What is mathematical practice?

“Is this data representative of mathematical practice?”

1. There is no single mathematical practice:

   - Inglis challenges ‘*Assumption of homogeneity’* with empirical studies into whether there is agreement between mathematicians on proof validity and appraisal

   - Diversity in mathematical practice recognised by conferences on mathematical cultures and practices (Larvor, 2016), by the ethnomathematics community, philosophical notions, such as mathematical style (Mancosu, 2009), etc.

2. The IMO is *a* mathematical practice:

   - It seems reasonable to assume that the Olympiad ‘culture’ may be regarded as background for a significant fraction of the world's professional pure mathematicians.

“Is this data representative of mathematical practice?”

Features of MPM culture:

- **Trust:** participants trust that the conjecture is a theorem and that there is a (findable) solution

- **Variety:** the number of collaborators and their range of mathematical experience, ability and knowledge may well be larger/wider than in other collaborative settings

- **Medium:** online fora mean that participants only communicate via typed comments in a very structured space - no diagrams, scribbles, gestures, intonation, ….

1. Patterns of communication

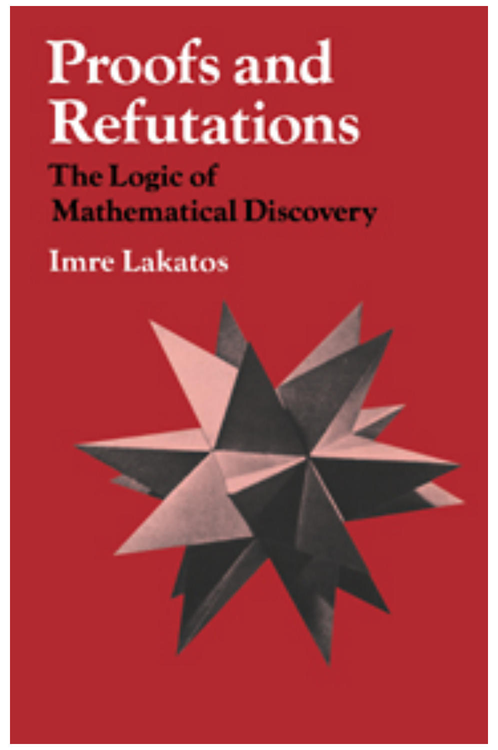
In collaboration with Alan Smaill (University of Edinburgh)
and Simon Colton (Imperial College London)

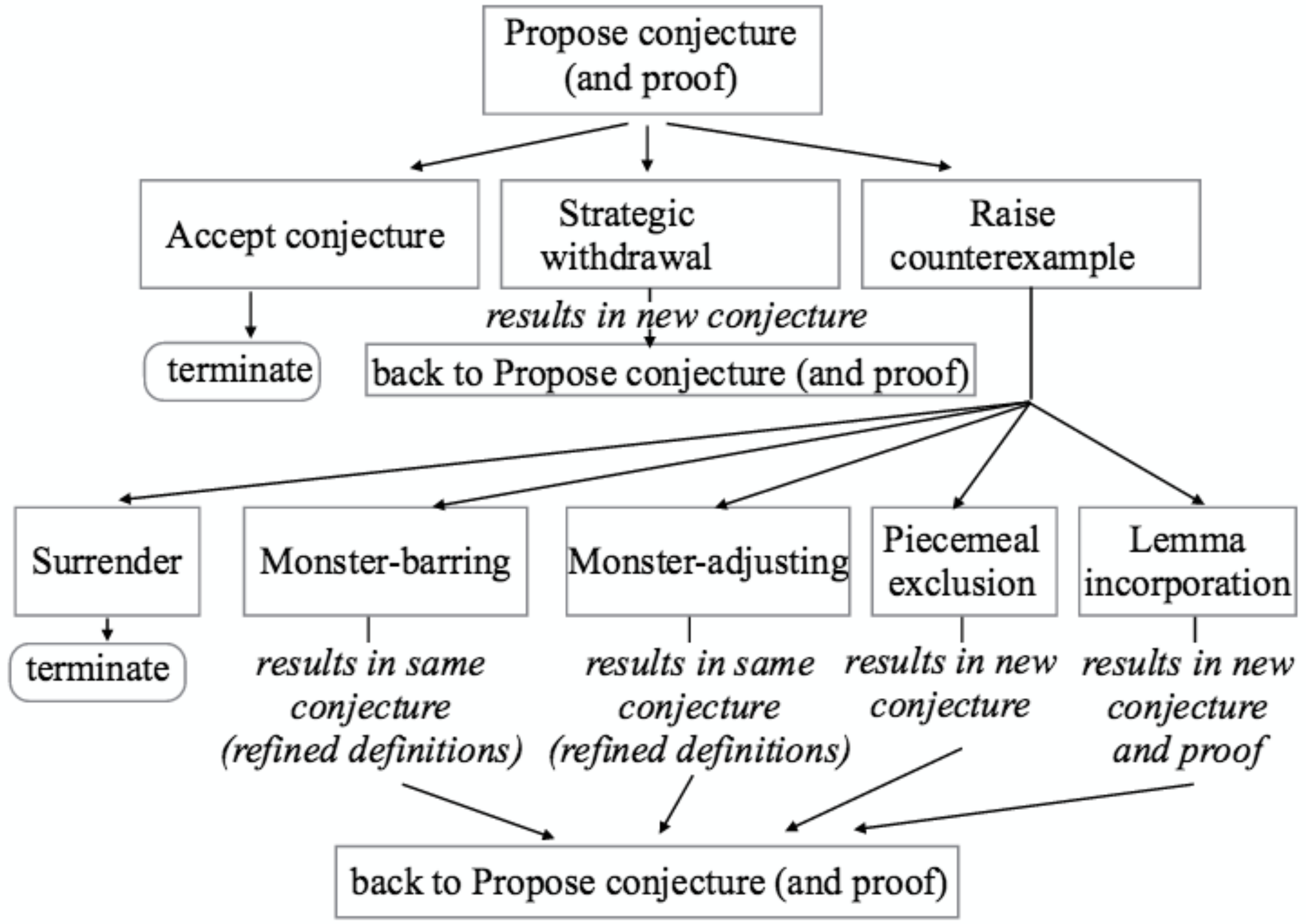
1. For all polyhedra, $V\text{-}E+F=2$

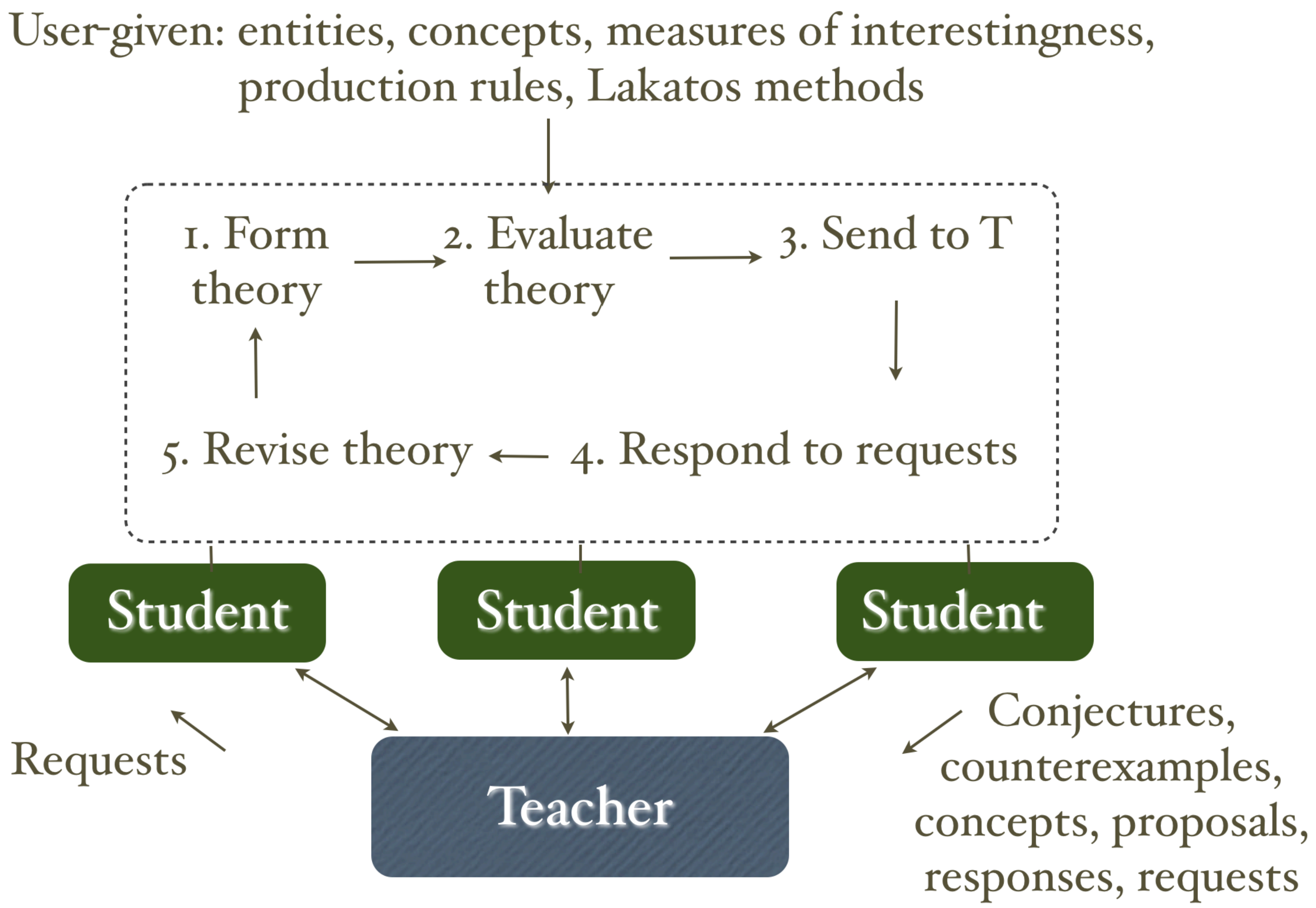
$\longrightarrow$ For all polyhedra, except those with cavities, $V\text{-}E+F=2$

2. For all polyhedra, $V\text{-}E+F=2$

$\longrightarrow$ For all convex polyhedra, $V\text{-}E+F=2$







1. Goldbach's conjecture:
All even numbers are the sum of two primes

$\longrightarrow$ All even numbers except 2 are the sum of two primes

2. All groups are Abelian

$\longrightarrow$ All self-inverse groups are Abelian

3. All integers have an even number of divisors

$\longrightarrow$ All non-squares have an even number of divisors

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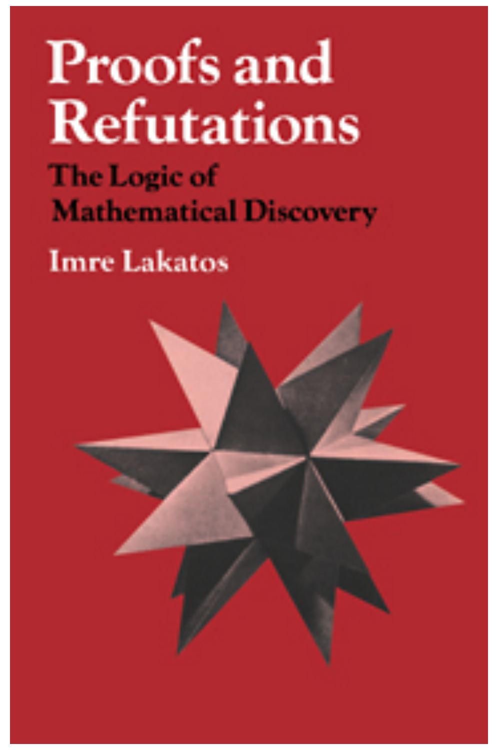
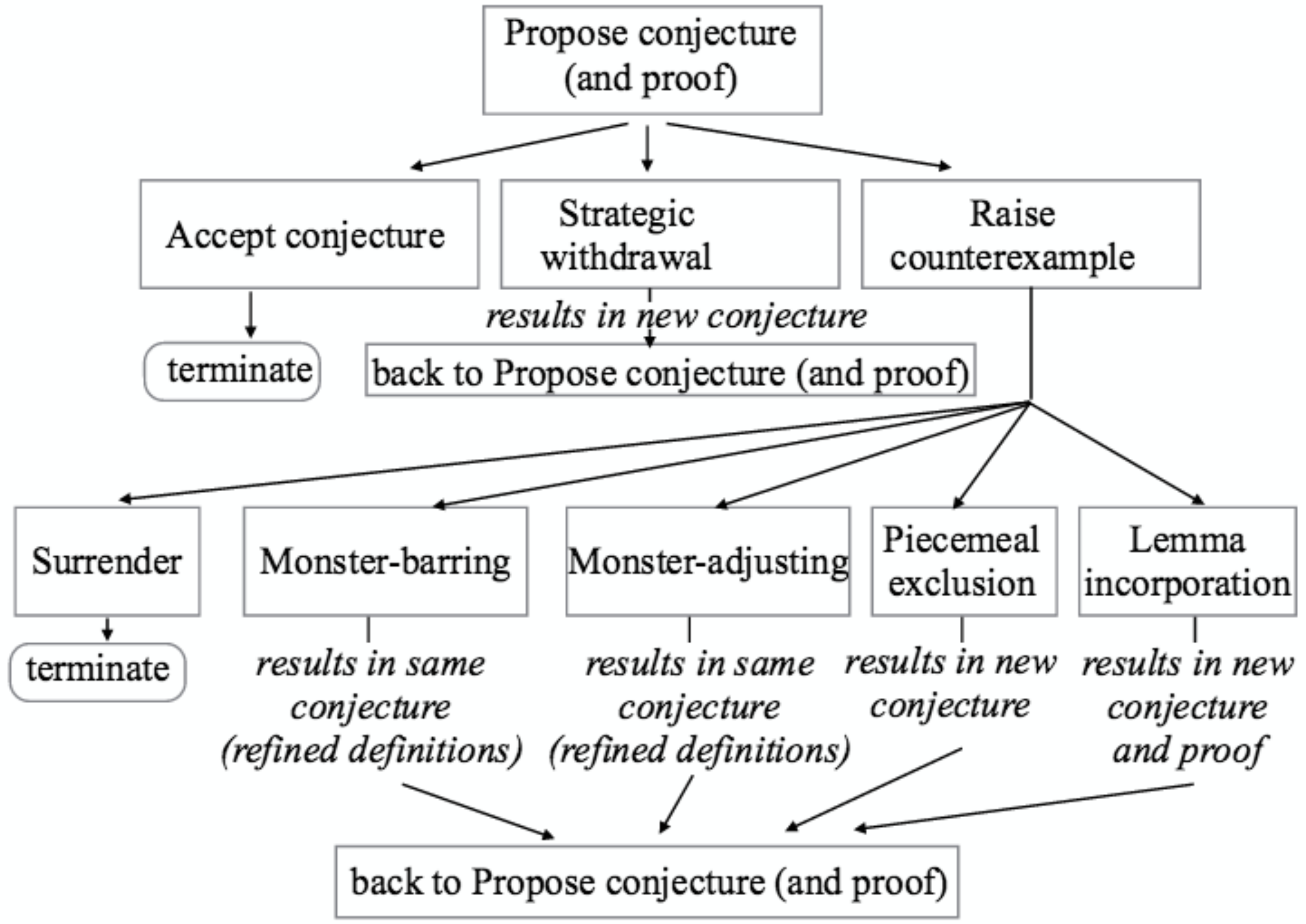
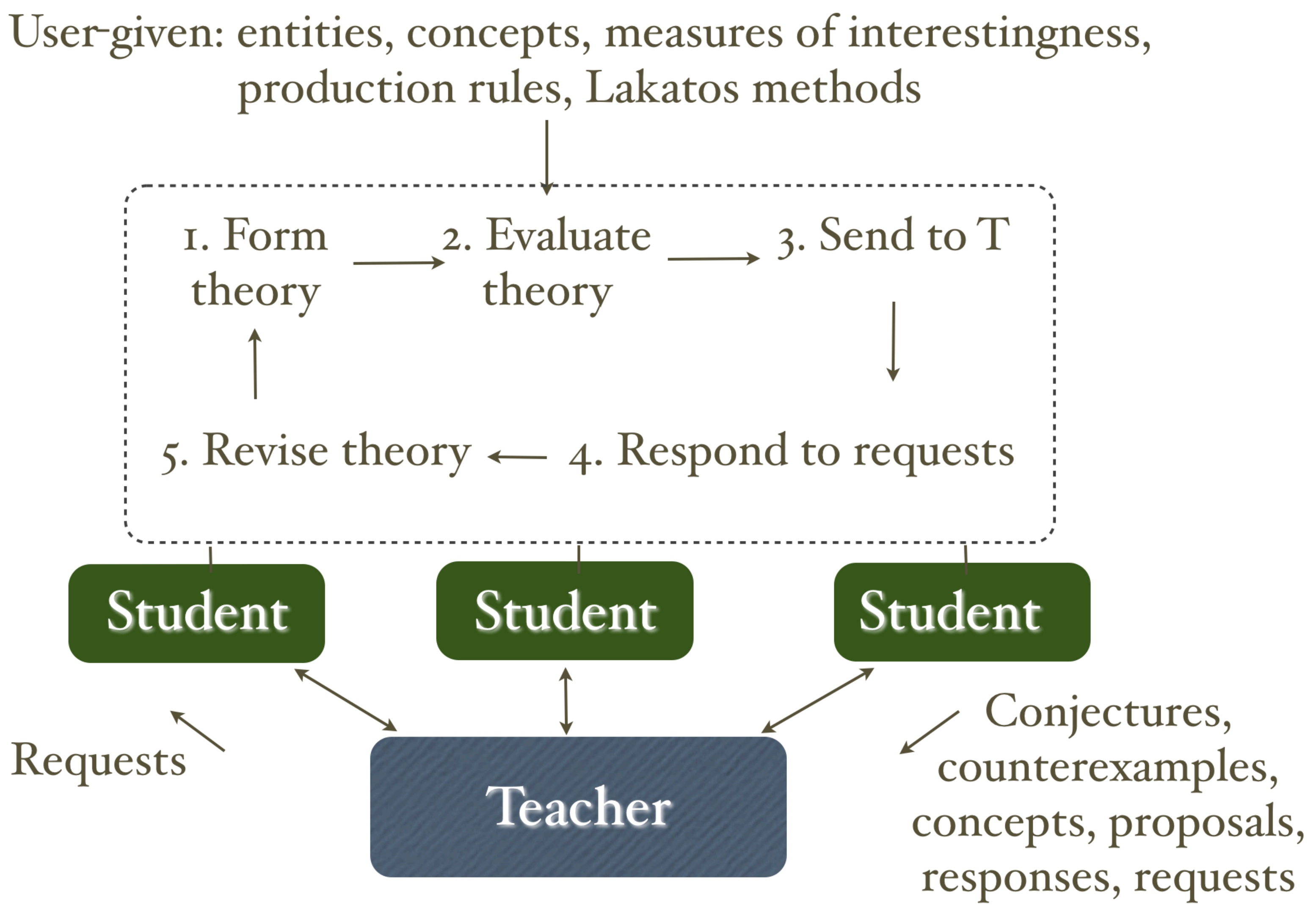




1. Is the system a faithful model of the theory? *(Evaluation of system)*
2. What has the computational perspective taught us? *(Evaluation of theory)*



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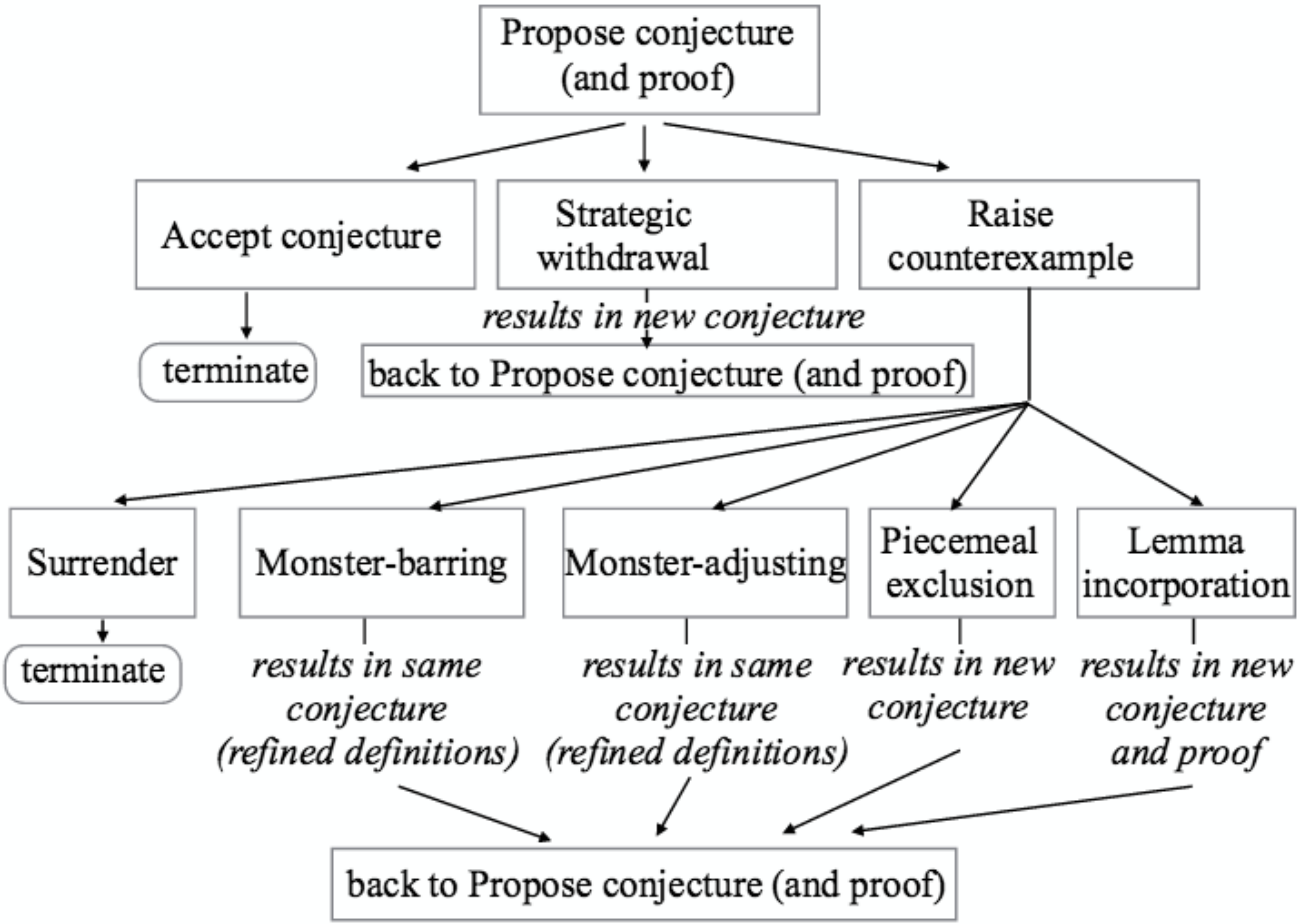
Propose conjecture (and proof)
Accept conjecture
Strategic withdrawal
Raise counterexample
terminate
results in new conjecture
back to Propose conjecture (and proof)
Surrender
Monster-barring
Monster-adjusting
Piecemeal exclusion
Lemma incorporation
terminate
results in same conjecture (refined definitions)
results in same conjecture (refined definitions)
results in new conjecture
results in new conjecture and proof
back to Propose conjecture (and proof)

HRL: Extending Simon Colton's HR System

Example interaction protocol

1. The teacher requests that the students work independently for twenty theory formation steps and then send an interesting conjecture.

2. The students comply and all send a conjecture.

3. The teacher sorts the conjectures into an agenda for discussion. It sends a request for modifications to the first conjecture on the agenda.

4. Each agent looks at the examples and counterexamples it has for the conjecture. If it has any counterexamples then it attempts to modify the conjecture and sends its modification.

5. The teacher sorts the modified conjectures into the agenda and sends a request for modifications to the next conjecture on the agenda.

Illustrative Example 1: set-up

Student 1: integers 0 – 10 and core concepts integers, divisors and multiplication. Propose to monster-bar if an entity is a culprit breaker. Agree a proposal to monster-bar if the entity is a counterexample to more than 15% of its conjectures.

Student 2: integers 1 – 10 and core concepts integers, divisors and multiplication. Set to use monster-barring as Student 1

The teacher requested the students to work independently for 20 theory formation steps and then send their best non-existence conjecture for discussion.

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Illustrative Example 1: results

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Student 2: There do not exist integers a, b such that $b + a = a$ and $a + b = a$

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Teacher: Does anyone have any counterexamples to this conjecture?

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Student 1: *All* of my entities (0-10) are counterexamples to this conjecture

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Teacher: How do you both respond to the counterexamples?

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Teacher: How do you both respond to the counterexamples?

Student 1: *[tests to see whether there was a single ‘culprit’ entity which is forcing all of its entities to be counterexamples, and found 0]* I propose to monster-bar the number 0.

Illustrative Example 1: results

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Student 1: *[tests to see whether there was a single 'culprit' entity which is forcing all of its entities to be counterexamples, and found 0]* I propose to monster-bar the number 0.

Student 2: *[finds that 0 breaks 63% of its conjectures]* I agree we should bar it

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Teacher: Okay, please down-grade 0 to a pseudo-entity in your theories.

Illustrative Example 2: set-up

Student 1: integers 1 – 10, core concepts integer, divisor and multiplication.

Student 2: integers 11 – 50, same core concepts

Student 3: integers 51 – 60, same core concepts

The teacher requested the students to work independently for 20 theory formation steps and then send their best implication conjecture for discussion.

Illustrative Example 2: results

Student 3: all integers have an even number of divisors

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Student 1: [1,4,9] are counterexamples

Illustrative Example 2: results

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Student 1: [1,4,9] are counterexamples

Student 2: [16, 25, 36, 49] are counterexamples.

Illustrative Example 2: results

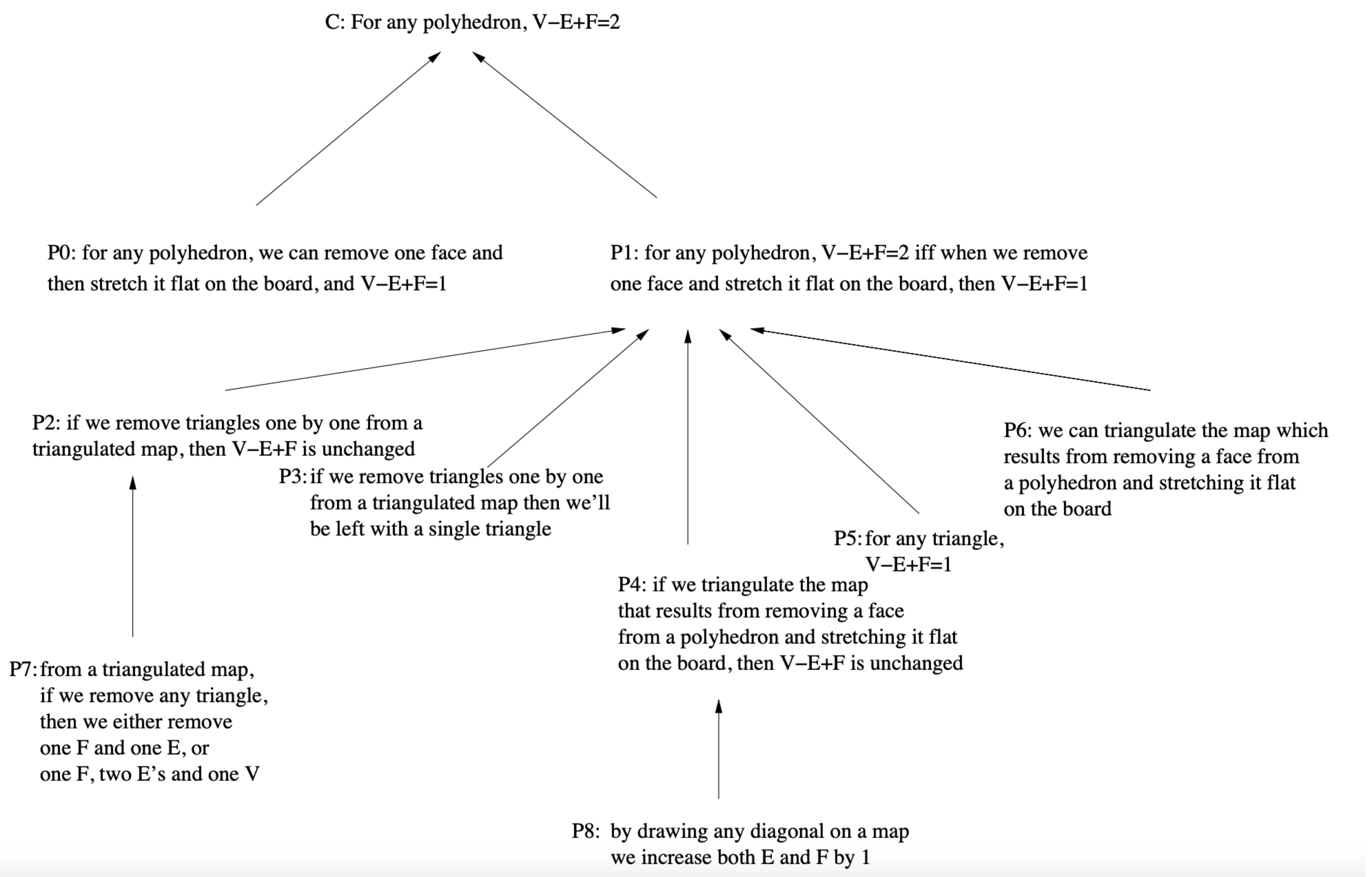
Student 3: all integers have an even number of divisors

Student 1: [1,4,9] are counterexamples

Student 2: [16, 25, 36, 49] are counterexamples.

Student 1: [finds concept of squares and formed the new concept non-squares]: We can modify **Student 3**'s conjecture to *all non-squares have an even number of divisors.*

Representing the proof

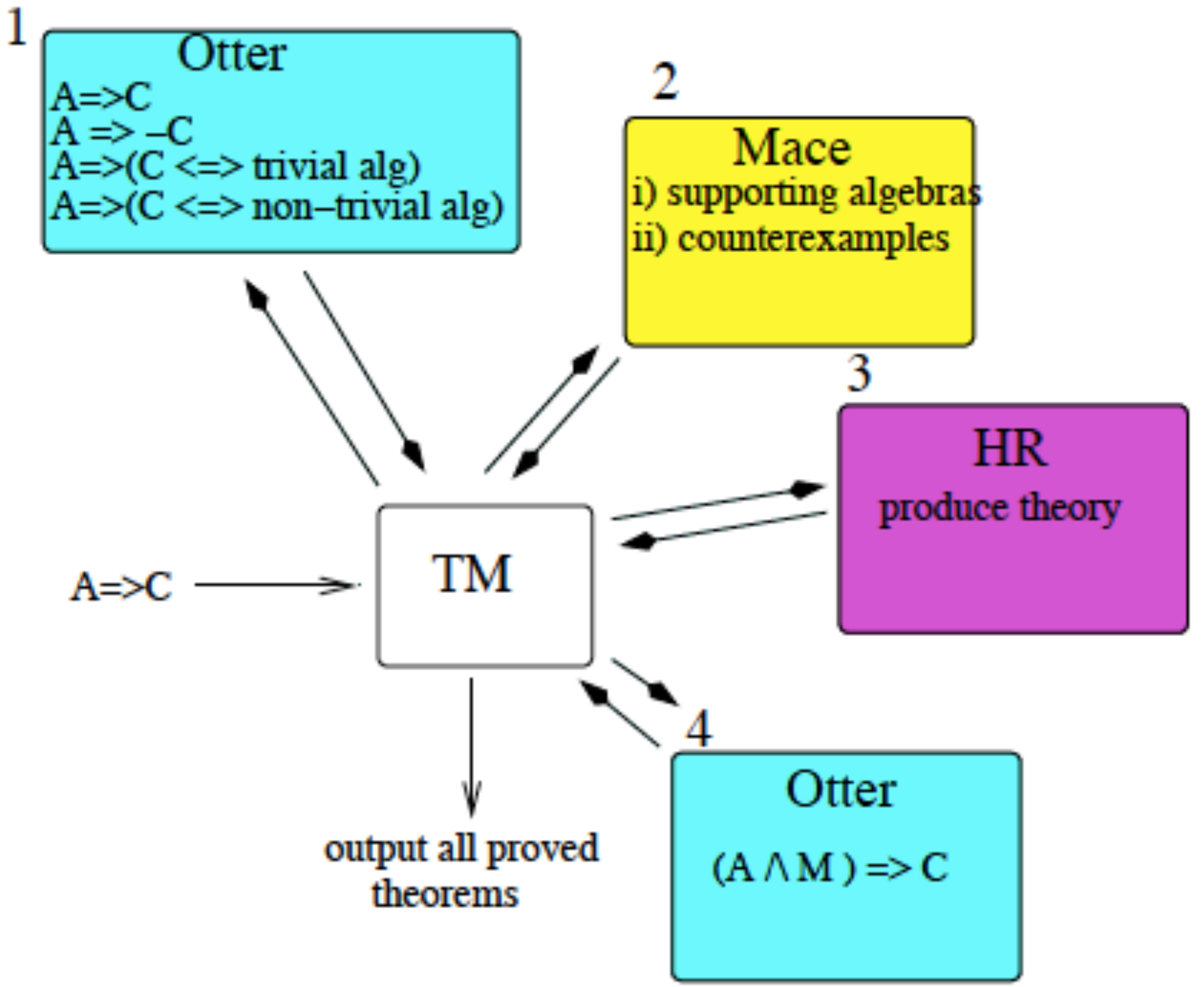


Challenges

- How do we know when we should surrender a conjecture?
- How can we computationally represent ill-defined or ambiguous concepts?
- When should we perform monster-barring?
- How can we apply exception-barring to different types of conjecture?
- How can we represent an informal proof in our system?
- How can a computer program uncover hidden assumptions in a proof?
- How can we formalise the surprise we feel when an example behaves in an unexpected manner in a proof?
- Given a counterexample, how can a computer program determine whether it is global or local?
- How can a computer program perform local/global/hidden lemma incorporation?

Two further interpretations of Lakatos's Proof and Refutations

Theorem Modifier (TM)



S. Colton and A. Pease "The TM System for Repairing Non-Theorems" Selected papers from the IJCAR'04 disproving workshop, Electronic Notes in Theoretical Computer Science, Volume 125(3). Elsevier, 2005

TM

- From TPTP library we invented 91 non-theorems. TM produced valid modifications for 83% of them, with an average of 3.1 modifications per non-theorem.
- Eg: Given non-theorem all groups are Abelian, TM produces all self-inverse groups are Abelian

Proofs and Refutations as a Dialogue Game

- **Theoretical model:** we interpret the informal logic of mathematical discovery proposed by Lakatos, a philosopher of mathematics, through the lens of dialogue game theory and in particular as a dialogue game ranging over structures of argumentation (locution rules, structural rules, commitment rules, termination rules and outcome rules).

- **Abstraction level:** we develop structured arguments, from which we induce abstract argumentation systems and compute the argumentation semantics to provide labelings of the acceptability status of each argument. The output from this stage corresponds to a final, or currently accepted proof artefact, which can be viewed alongside its historical development.

- **Computational model:** we show how each of these formal steps is available in implementation

A Pease, J Lawrence, K Budzynska, J Corneli, Reed. *Lakatos-style collaborative mathematics through dialectical, structured and abstract argumentation.* Artificial Intelligence. Vol 246, 2017. pp181-219.

Are Lakatos's patterns of communication found in the real world?

Example 1

20 July, 2009 at 6:51 am
Haim

~~2~~ 5. The following reformulation of the problem may be useful:
Show that for any permutation s in Sn, the sum a_s(1)+a_s(2)…+a_s(j) is not in M for any j=<n.

Now, we may use the fact that Sn is "quite large" and prove the existence of such permutation with some kind of a pigeonhole-ish principle

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P is equivalent to P’

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20 July, 2009 at 7:01 am
Dave

~~1~~ 10.

Addressing Michael Lugo: I think he means number just your own comments, and then address a (person,number) pair. *[Actually, I was proposing a global numbering system; I'll try to fix it up now. But the (author, number) pair approach would also have worked, except perhaps for anonymous comments. -T]*

Addressing Haim(~~2~~ 5):

That's pretty strong; all you need is that there exists a permutation where that is true. And it doesn't work; there are numbers $a_1,a_2,\ldots,a_n$ and sets M of $n-1$ points such that, for instance, $a_1 \in M$. Then any permutation starting with a_1 would not satisfy your conjecture for $j=1$.

But, just looking for *one* permutation that satisfies $a_s(1)+a_s(2)…+a_s(j) \not \in M$ for any $j \leq n$ (which is basically the statement of the theorem), could lend itself well to induction. In other words, use the fact that for every subset $M' \subset M$ of size j not containing $a_s(1)+a_s(2)…+a_s(j)$, there is a way to permute those j numbers to avoid M'.

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0 0 Rate This

P is not equivalent to P'. Here is a counterexample.

P is equivalent to P''

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P is not equivalent to P’. Here is a counterexample.

P is equivalent to P”

0 0 Rate This

20 July, 2009 at 7:10 am

Haim

12. Addressing Dave:

Sorry, indeed I meant: “Show that for *one* permutation...”

0 0 Rate This

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0 0 Rate This

P is not equivalent to P’. Here is a counterexample.

P is equivalent to P’’

20 July, 2009 at 7:10 am
Haim

12. Addressing Dave:

Sorry, indeed I meant: “Show that for *one* permutation…”

0 0 Rate This

I surrender - it is not the case that P is equivalent to P’

Are Lakatos's patterns of communication found in the real world?

Example 2

Say there are four points: an equilateral triangle, and then one point in the center of the triangle. No three points are collinear.
It seems to me that the windmill can not use the center point more than once! As soon as it hits one of the corner points, it will cycle indefinitely through the corners and never return to the center point.
I must be missing something here…

0 0 Rate This

Comment by Jerzy — July 19, 2011 @ 8:17 pm

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Isn't X a counterexample?

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This isn't true – it will alternate between the centre and each vertex of the triangle.

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1 0 Rate This

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Ah yes, I'll revise my concept definition

1 0 Rate This

Comment by Jerzy — July 19, 2011 @ 8:31 pm

Lessons for ATP

GAMMA: Yes. (6) and (7) are not growth, but degeneration! Instead of going on to (6) and (7), I would rather find and explain some *exciting* new counterexample![1]

ALPHA: You may be right after all. But who decides *where* to stop? Depth is only a matter of taste.

GAMMA: Why not have mathematical critics just as you have literary critics, to develop mathematical taste by public criticism? We may even stem the tide of pretentious trivialities in mathematical literature.[2]

SIGMA: If you stop at (5) and turn the theory of polyhedra into a theory of triangulated spheres with *n* handles, how can you, if the need arises, deal with trivial anomalies like those explained in (6) and (7)?

MU: Child's play!

```
if((current_request.motivation.attempted_method).equals("monster-barring")
   && !(current_request.motivation.entity_under_discussion==null))
  {

Entity entity = current_request.motivation.entity_under_discussion;
//into monster-barring. evaluating the proposal

//extra bit -- lakatos variable for mb -- need to put this elsewhere too
boolean use_breaks_conj_under_discussion = hr.theory.lakatos.use_breaks_conj_under_discussion;
if(use_breaks_conj_under_discussion)
{
  Conjecture conj = current_request.motivation.conjecture_under_discussion;
  if(conj.counterexamples.isEmpty())
    {
      String mb_vote = "accept proposal to bar entity";
      response.response_vector.addElement(mb_vote);
    }
  if(!(conj.counterexamples.isEmpty()))
    {
      String mb_vote = "reject proposal to bar entity";
      response.response_vector.addElement(mb_vote);
    }
}
```

- It is possible to give a computational reading of Lakatos's Proofs and Refutations, including very "human", messy aspects such as ambiguity, hidden assumptions, surprise, … This offers flexibility to ATP and suggests ways of bringing it closer to "human" mathematics

Lessons for ATP

- We can fine-tune the methods and test parameters and methods to see which ones result in interesting theories. Eg, how to distribute data between the students; how many independent work phases, how long; how and when to perform the methods
- A computational model also allows us to evaluate extended computational theory of mathematical discovery according to philosophically interesting criteria such as generality, explanatory power and non-circularity
- Collaborative patterns such as Lakatos's Proofs Refutations can be found in real world mathematics, although can be less useful than he described
- Errors and ambiguities can be productive (but might not be)

2. What do mathematicians talk about?

2. What do mathematicians talk about?

1. Broad stroke analysis
2. Fine-grained analysis

2. What do mathematicians talk about?

1. Broad stroke analysis
2. Fine-grained analysis

What do mathematicians talk about?

Question 2 of the 2011 IMO.

Tao posted the problem at 8pm on July 19th, 2011, having posted in advance that he would do so.

The relevant websites are the *research thread* for the problem solving process, a *discussion thread* for meta-discussion about the project, and a *wiki page* for a summary of the problem and discussion.

Solved over a period of 74 minutes by 27 participants through 174 comments on 27 comment threads.

Course-grained analysis to develop a typology of comments

A. Pease and U. Martin. *Seventy four minutes of mathematics: An analysis of the third mini-polymath project.* In Proc AISB Symp on Mathematical Practice and Cognition II pages 19-29, 2012.

Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A **windmill** is a process that starts with a line l going through a single point P ∈ S. The line rotates clockwise about the pivot P until the first time that the line meets some other point Q belonging to S. This point Q takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of S. This process continues indefinitely. Show that we can choose a point P in S and a line l going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.

A typology of comments

Concepts:

Since the points are in general position, **you could define "the wheel of p"** w(p) to be radial sequence of all the other points p'!=p around p. Then, every transition from a point p to q will "set the windmill in a particular spot" in q. This device tries to clarify that the new point in a windmill sequence depends (only) on the two previous points of the sequence.

A typology of comments

Examples:

If the points form a **convex polygon**, it is easy.

A typology of comments

Conjectures:

One can start with any point (since every point of S should be pivot infinitely often), the direction of line that one starts with however matters!

Perhaps even the line does not matter! Is it possible to prove that any point and any line will do?

A typology of comments

Proof:

The first point and line P_0, l_0 cannot be chosen so that P_0 is on the boundary of the convex hull of S and l_0 picks out an adjacent point on the convex hull.
Maybe the strategy should be to take out the convex hull of S from consideration; follow it up by induction on removing successive convex hulls.

A typology of comments

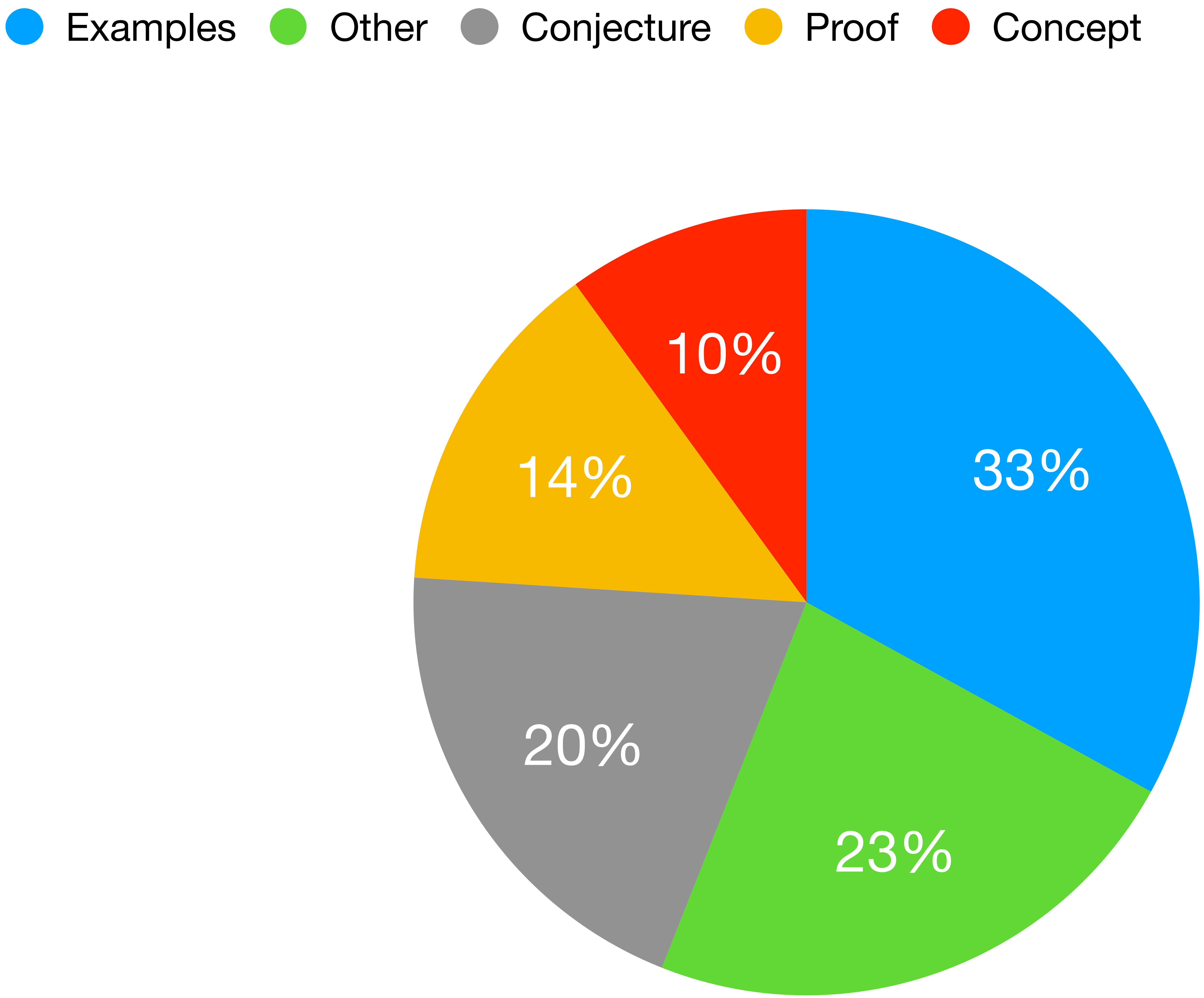
Other:

I think that is a good start, thanks Varun!

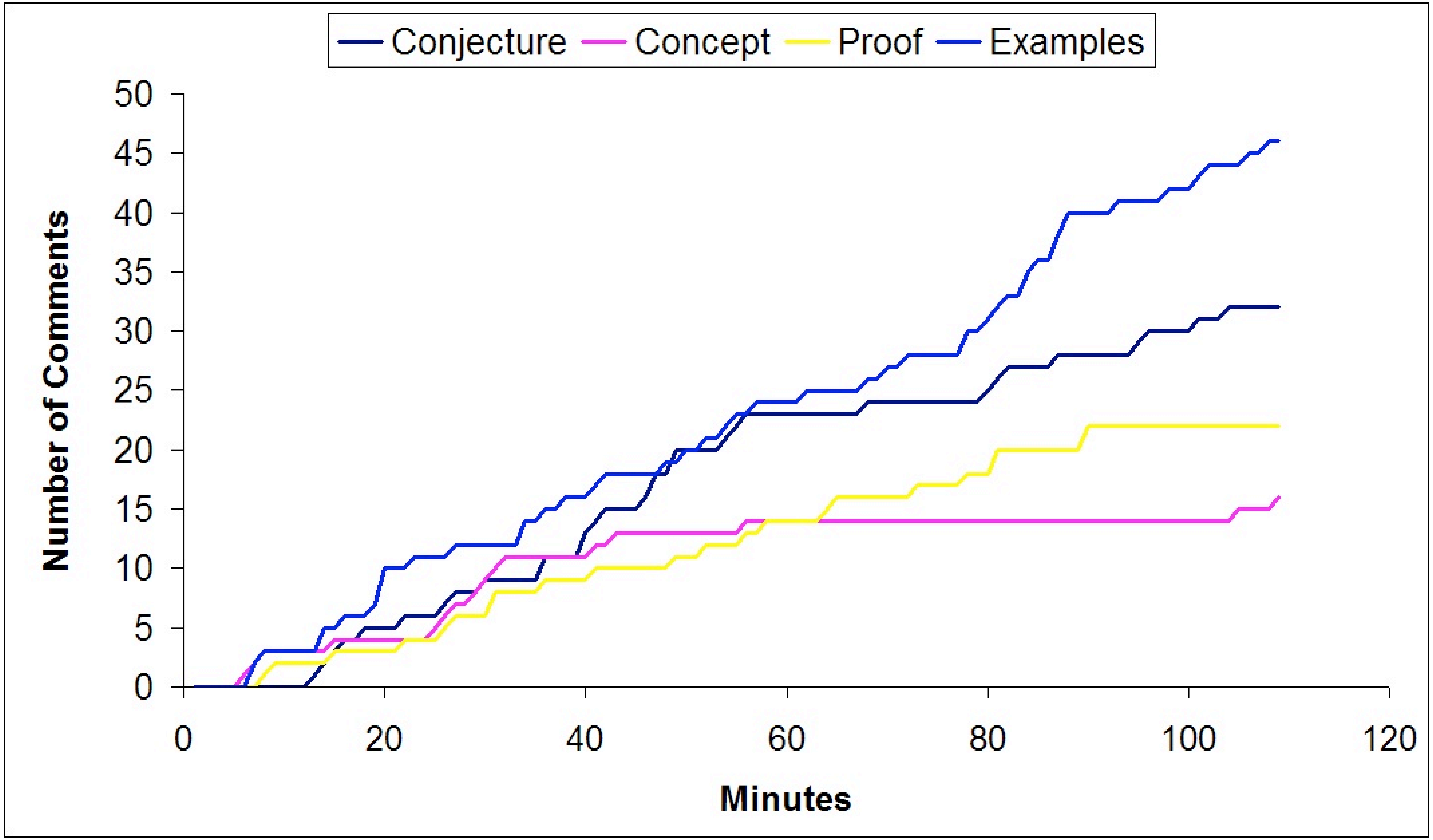
@Thomas @Seungly @Haggai Thank you all for your examples. I haven't understood them fully yet; I'll think about them for some time and get back if I have questions.

Yes, it seems to be a correct solution!

A typology of comments



A typology of comments



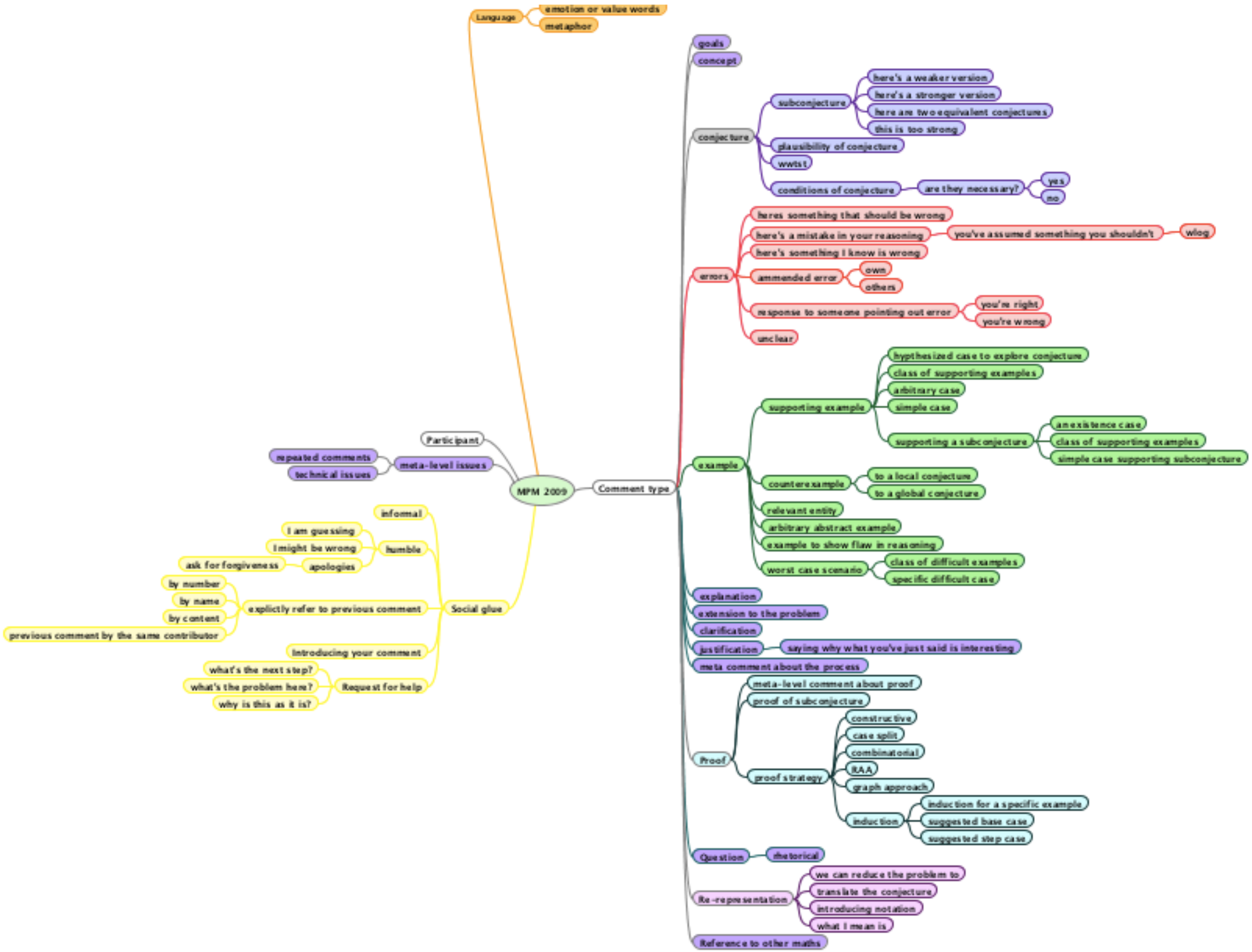
2. What do mathematicians talk about?

1. Broad stroke analysis
2. Fine-grained analysis

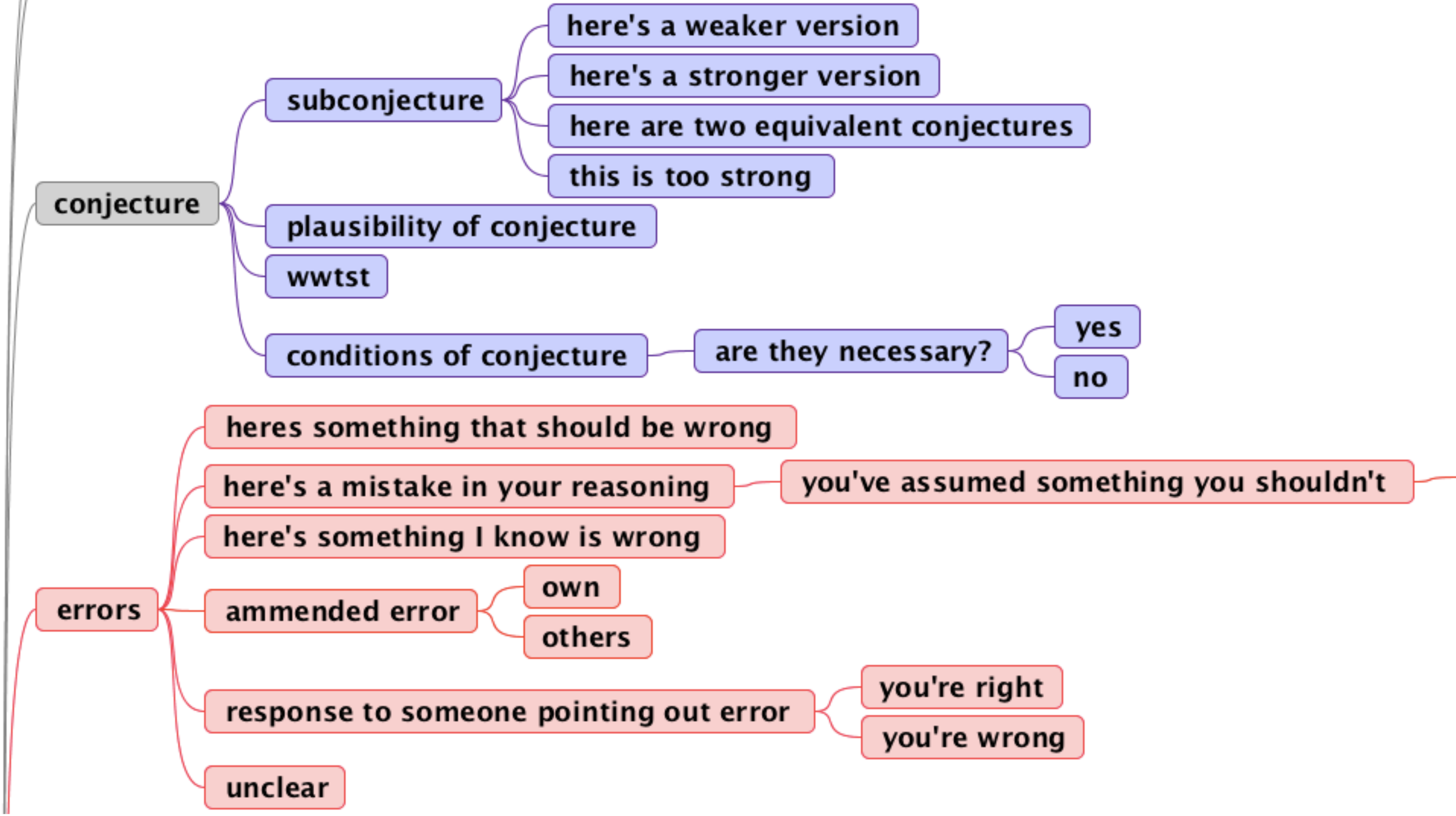
A fine-grained analysis

- Similar - more in depth - analysis of the first MPM, posted in 2009
- Used software for Grounded Theory “dedoose" to create hierarchies of tags
- Tagged 559 excerpts

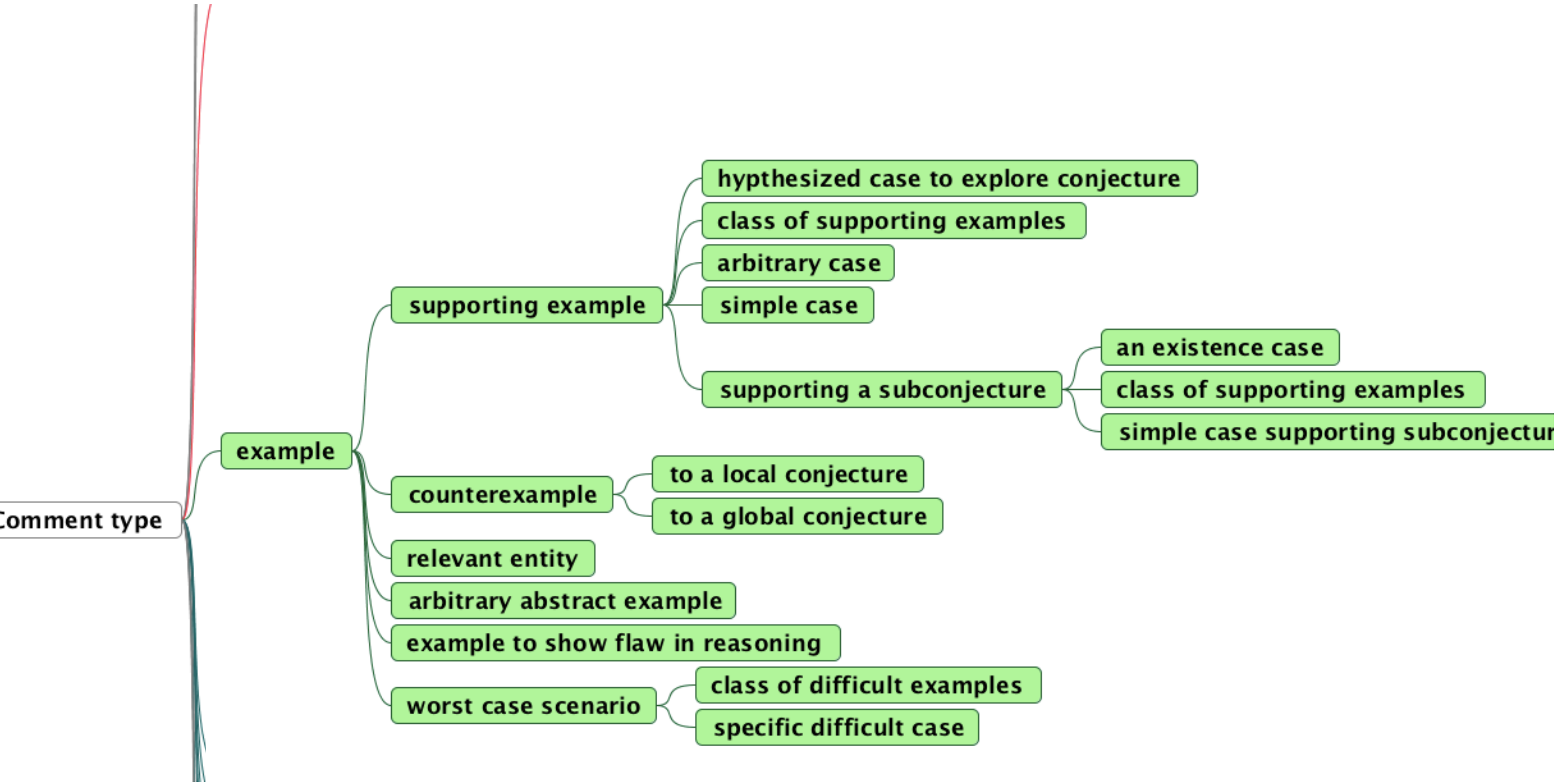
dedoose
Great Research Made Easy
mpm1 | Logout | Account
Home
Codes
Media
Excerpts
Descriptors
Analyze
Memos
Training
Security
Data Set
Back
Projects
Document: mpm1-2009
Line #'s Memos RTL
Added: 12/04/2013 Creator: alisonp Excerpts: 559
Memos: 0
Descriptors: 0
gowers
54 60. Another small case. Let's take a_i=i for i=1,2,3,4. So we're trying to get to 10 in steps of 1,2,3,4 and there are three landmines.
If there's a landmine on any of 1,2,3,4, then by 47 (@liuxiaochuan) they must be on 4, or 4 and 3, or 4 and 3 and 2. In the third case we can go to 1 and then to 5, and then we're done by induction (two steps and zero obstacles, so perhaps induction was a bit of a sledgehammer). If there are obstacles on 4 and 3, then induction is more appropriate — we can either get to 5 in two steps and are then done, or there's an obstacle at 5, in which case we can go 2,6,7,10. If there's just an obstacle at 4, things get harder, since then we need to know what goes on after 4. But then we can cheat and say that at least one number between 6 and 9 is an obstacle so we can run things in reverse. The only case not covered is then when the obstacles are at 4,5,6.
That was still a rather ugly case-by-case argument, but it serves to confirm a sense that the difficult case is when the obstacles are not near the end points.
I'd like to try to find an argument along the following lines. Order the step sizes as a_1<\dots<a_n. Now let's try two paths. The first is where you take the steps in increasing order of size, and the second is where you take it in decreasing order. Now look at where you are half way through this process. Suppose that in the first case you have passed well under half the obstacles and in the second case you've passed well over half. Then it should be possible to move from one extreme to the other and find a permutation where you've passed more or less exactly half. (Actually, of course, the hypothesis here doesn't have to hold, but this is just meant to give the flavour of some kind of argument.) And then there might be a hope of … hmm … I'm still trying to find that elusive jump over two obstacles that takes place at exactly the right time.
Subquestion. If a_i=i and you have n-1 consecutive obstacles, what's the neatest proof that you must be able to get to the last non-obstacle without using a_n? (I don't think it's hard to prove it, but it would be good to have something that had a hope of generalizing.)
1
0
Rate This
20 July, 2009 at 12:03 pm
gowers
63. Re 54. Your analysis shows that it is possible for the gaps all to be at least a_n-1. Just let all the a_n be very large and roughly equal to a. Then the sum is approximately equal to na, so if we start near 0 and end near na, then we can get gaps of size about na/(n-1), which is bigger than a_n. So my proposal runs into difficulty. Not sure how big a difficulty it is though.
1
0
Font Size: 0
Prev Excerpt
Next Excerpt
Selection: (34866-34866)
Create Excerpt
Selection Info
mpm1-2009 (34840-34999)
meta-level comment abo...
mpm1-2009 (34864-34868)
emotion or value words
Codes
comment type
clarification
concept
conjecture
errors
example
explanation
extension to the problem
goals
justification

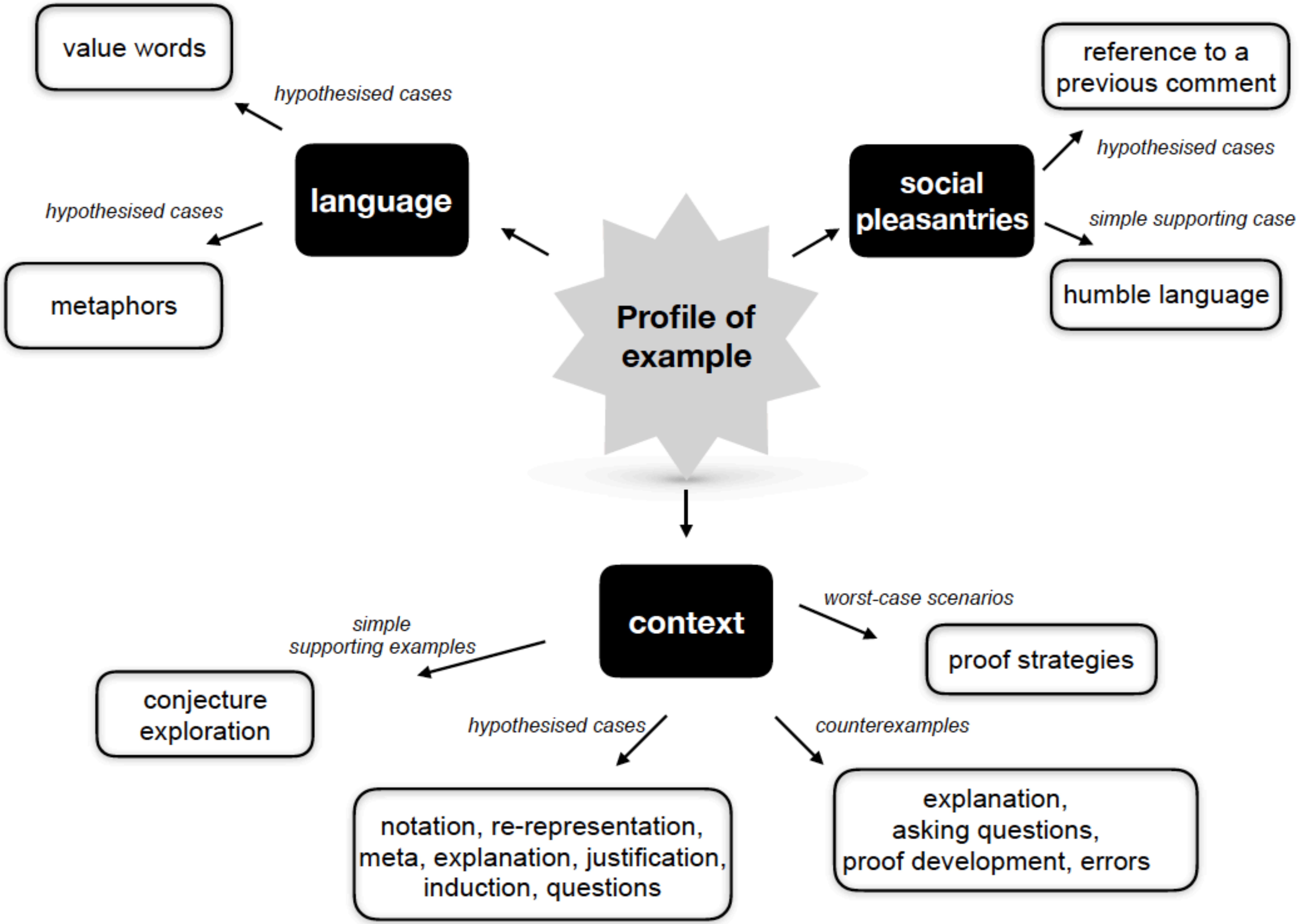


MPM 2009
Language
emotion or value words
metaphor
Participant
meta-level issues
repeated comments
technical issues
Social glue
informal
humble
I am guessing
I might be wrong
apologies
ask for forgiveness
explictly refer to previous comment
by number
by name
by content
previous comment by the same contributor
Introducing your comment
Request for help
what's the next step?
what's the problem here?
why is this as it is?
Comment type
goals
concept
conjecture
subconjecture
here's a weaker version
here's a stronger version
here are two equivalent conjectures
this is too strong
plausibility of conjecture
wwtst
conditions of conjecture
are they necessary?
yes
no
errors
heres something that should be wrong
here's a mistake in your reasoning
you've assumed something you shouldn't
wlog
here's something I know is wrong
ammended error
own
others
response to someone pointing out error
you're right
you're wrong
unclear
example
supporting example
hypthesized case to explore conjecture
class of supporting examples
arbitrary case
simple case
supporting a subconjecture
an existence case
class of supporting examples
simple case supporting subconjecture
counterexample
to a local conjecture
to a global conjecture
relevant entity
arbitrary abstract example
example to show flaw in reasoning
worst case scenario
class of difficult examples
specific difficult case
explanation
extension to the problem
clarification
justification
saying why what you've just said is interesting
meta comment about the process
Proof
meta-level comment about proof
proof of subconjecture
proof strategy
constructive
case split
combinatorial
RAA
graph approach
induction
induction for a specific example
suggested base case
suggested step case
Question
rhetorical
Re-representation
we can reduce the problem to
translate the conjecture
introducing notation
what I mean is
Reference to other maths



conjecture
subconjecture
here's a weaker version
here's a stronger version
here are two equivalent conjectures
this is too strong
plausibility of conjecture
wwtst
conditions of conjecture
are they necessary?
yes
no
errors
heres something that should be wrong
here's a mistake in your reasoning
you've assumed something you shouldn't
here's something I know is wrong
ammended error
own
others
response to someone pointing out error
you're right
you're wrong
unclear


Comment type
example
supporting example
hypthesized case to explore conjecture
class of supporting examples
arbitrary case
simple case
supporting a subconjecture
an existence case
class of supporting examples
simple case supporting subconjectur
counterexample
to a local conjecture
to a global conjecture
relevant entity
arbitrary abstract example
example to show flaw in reasoning
worst case scenario
class of difficult examples
specific difficult case


Profile of example
language
hypothesised cases
value words
hypothesised cases
metaphors
social pleasantries
hypothesised cases
reference to a previous comment
simple supporting case
humble language
context
simple supporting examples
conjecture exploration
worst-case scenarios
proof strategies
hypothesised cases
notation, re-representation, meta, explanation, justification, induction, questions
counterexamples
explanation, asking questions, proof development, errors

Lessons for ATP

- Mathematicians talk about many aspects other than proof: concepts, conjectures, examples, social mechanisms, explanations
- These are introduced at different points in a proof attempt, for different reasons

3. How do mathematicians explain things?

Source Material: Mini-Polymath Data

Year	IMO	Timeline	Comments/Words (before solution)	Participants
2009	Q6	**Start:** July 20, 2009 @ 6:02 am **Solution:** 21 July, 2009@ 11:16 am **End:** August 15, 2010 @ 3:30 pm	356/32,430 (201)	81-100
2010	Q5	**Start:** July 8, 2010 @ 3:56 pm **Solution:** July 8, 2010 @ 6:24 pm **End:** July 12, 2012 @ 6:31 pm	128/7,099 (75)	28
2011	Q2	**Start:** July 19, 2011 @ 8:01 pm **Solution:** July 19, 2011 @ 9:14 pm **End:** October 17, 2012 @ 3:25 pm	151/9,166 (70)	43-56
2012	Q3	**Start:** July 12, 2012 @ 10:01 pm **Solution:** July 13, 2012 @ 7:53 pm **End:** August 22, 2012 @ 3:27 pm	108/10,097 (79)	43-48
TOTAL:			742/58,792	185 - 221

Methodology

Explanation-indicator approach (EIA): Close content analysis of an complete search over four mathematical conversations, based on the presence of explanation indicators: *“since”; “because”; “, as”* (premise); *“thus”; “therefore”; “, so”* (conclusion); *“expla*”; “underst*”* (explanation).

Stage 1: Automated complete indicator search

Stage 2: Close content analysis of each indicator instance - the entire comment and surrounding comments, using cues from these to try to determine whether explanation played a role

Stage 3: Close content analysis according to our dimensions of explanation

Methodology

Random comment approach (RCA): designed to pick out explanations which might not contain any of our explanation indicators. We analysed a random 10% of our corpora of 742 comments. The analysis followed similar stages to the EIA:

Stage 1: Select a comment at random and discount it if it contains an explanation indicator

Stage 2: Close content analysis of the entire comment and comments around it, using cues from these to try to determine whether explanation played a role

Stage 3: Close content analysis according to our dimensions of explanation

Methodology

Contextual considerations in Stage 3: From sentence, comment and surrounding comments, we:

- identify the explanans (the explanation) and the explanandum (the phenomenon to be explained)

- consider whether there is a corresponding why-question (possibly implicit)

- consider whether there is a clear difference of level (general or specific) between the explanandum and explanans which might be seen in terms of a unifying feature.

- ask whether the keyword occurred in the context of object-level, meta-level or deep (additional context) explanation

- consider what the context of the proof is, what sort of thing is being explained, and what sort of explanation is offered

Individual context

Abilities (what can/can't we do). [Keywords: difficulty, hard, do] Examples: We can only almost do X; We can do X; We must be able to do X; X is always possible; X might not be the hardest bit; We can fix X in this way; The difficult bit might be X.

Knowledge (what do/don't we know). [Keywords: know, plausible, mistake, wrong, assume, obvious, suppose] Examples: We don't know X; X is plausible; X is wrong; X is a mistake.

Understand (what do/don't we understand). [Keywords: understand] Examples: Why is this a contradiction?

Value/goals (what do/don't we want). [Keywords: want, goal, need, help, problem, target, useful] Examples: X is a good idea; We want to do X; X will achieve our goal; We need to know X; X will help us in this way.

Mathematical context

Initial problem. Examples: The initial problem is harder if X; The initial problem is hardest when X; Condition X is necessary for the initial problem.

Proof (approach). Examples: X is not a useful approach; Approaches X and Y might be the same; Approach X might not work; If we can do X then we have a complete proof.

Assertions. Examples: There is only one of type X; x is not in set X; Y is a subset of X; If we do X then we'll get Y; There must always exist X that satisfies condition Y.

Specific cases/instances. Examples: Things get harder in case X; There will always exist instance X that satisfies condition Y; The problem works in instance X; other cases X and Y are trivial; Case X might be a problem.

Arguments. Examples: Let us suppose X. Then Y.

Representation. Examples: there are many ways to write X; by reducing the problem to X.

Property. Examples: X has this property; X might not be unique; X doesn't have this property; X might have this property.

Social context

Type of dialogue	Initial situation	Participant's goal	Goal of dialogue
Persuasion	Conflict of opinions	Persuade other party	Resolve or clarify issue
Inquiry	Need to have proof	Find and verify evidence	Prove (disprove) hypothesis
Negotiation	Conflict of interests	Get what you most want	Reasonable settlement that both can live with
Information-seeking	Need information	Acquire or give information	Exchange information
Deliberation	Dilemma or Practical choice	Co-ordinate goals and Actions	Decide best available course of action
Eristic	Personal conflict	Verbally hit out at opponent	Reveal deeper basis of conflict

Walton (2000) p336

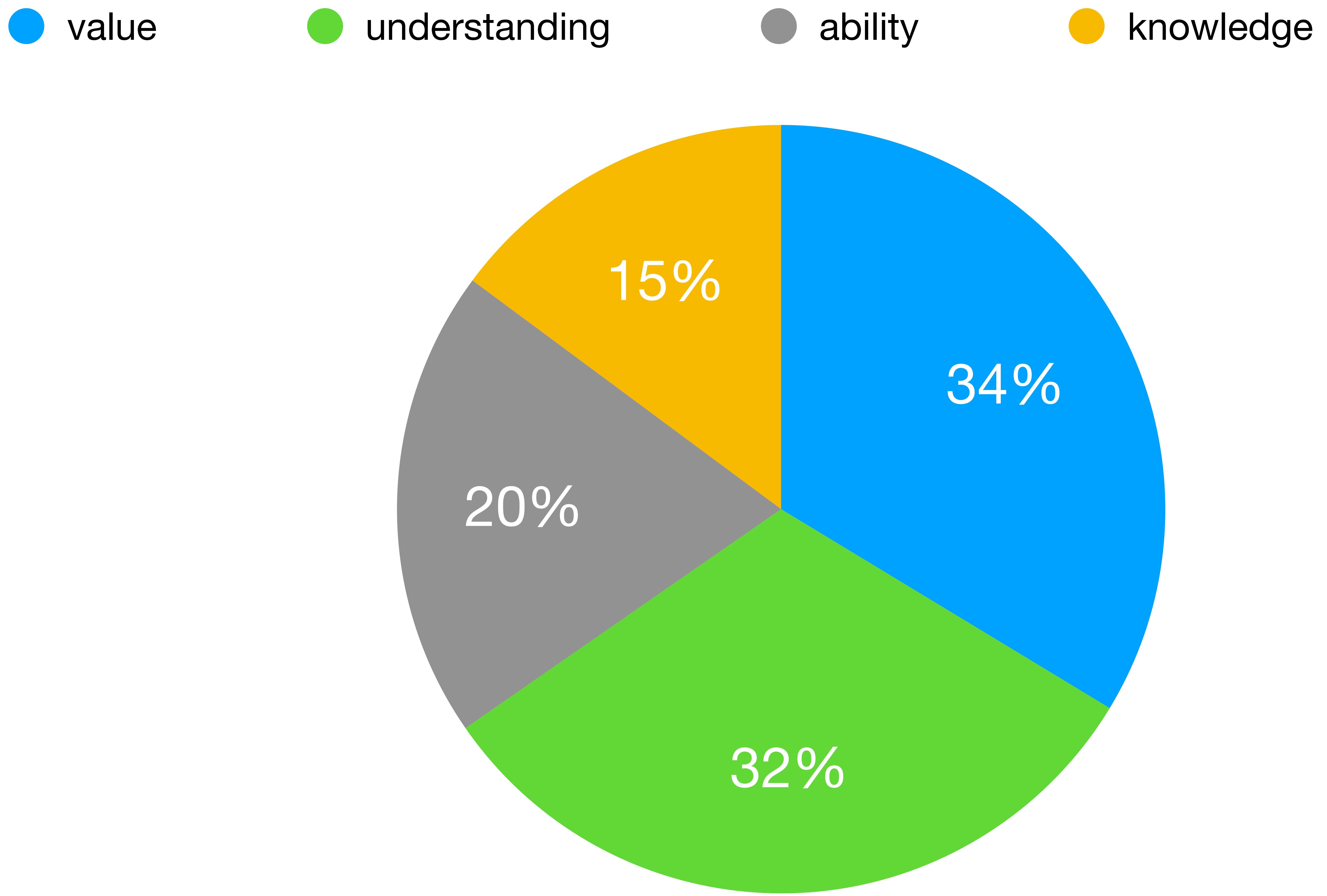
Findings

- **EIA:** On average, 33% of comments contained one indicator, of which we classified 72% as relating to explanation. This gives 23% of the whole conversation as relating to explanation.
- **RCA:** Of the remaining 67% of the conversation which did not contain an indicator, we classified 21% of a random sample as relating to explanation.This gives a further 14% of the whole conversation.
- This gives a combined total of 37%. Explanation here formed an important part of the mathematical conversation.

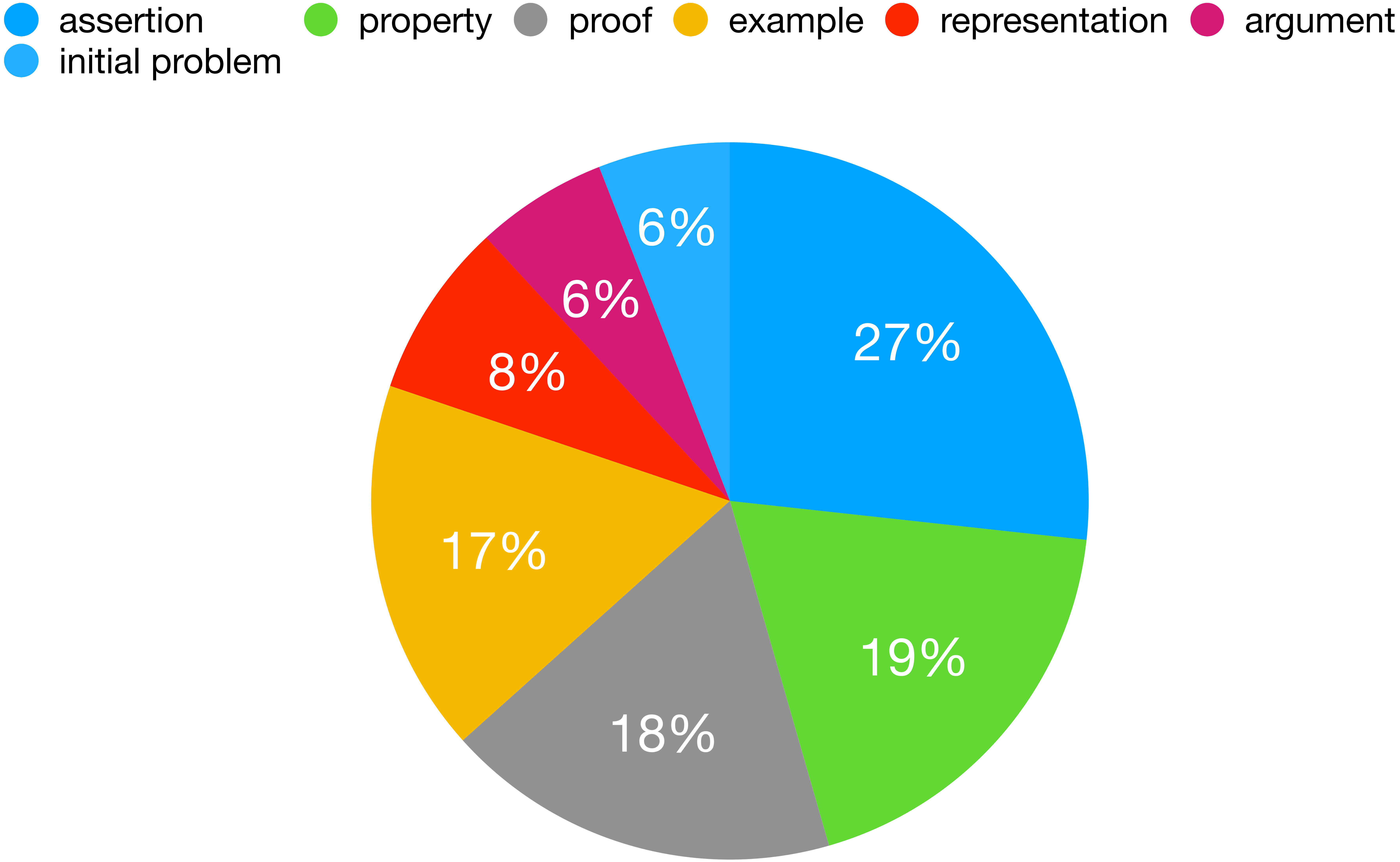
Findings

		raw		%	
Hypothesis	Labelling	EIA	RCA	EIA	RCA
H2	Answers to why questions	174	13	99%	100%
	Not answers to why questions	2	0	1%	0%
H3	Primarily an appeal to a higher level	4	3	2%	23%
	Not primarily an appeal to a higher level	172	10	98%	77%
H4	Trace explanation	121	8	69%	62%
	Strategic explanation	42	4	24%	31%
	Deep explanation	11	1	6%	8%
	Neither trace, strategic, nor deep	2	0	1%	0%

The Individual Context:
Does the explanation use purposive words?

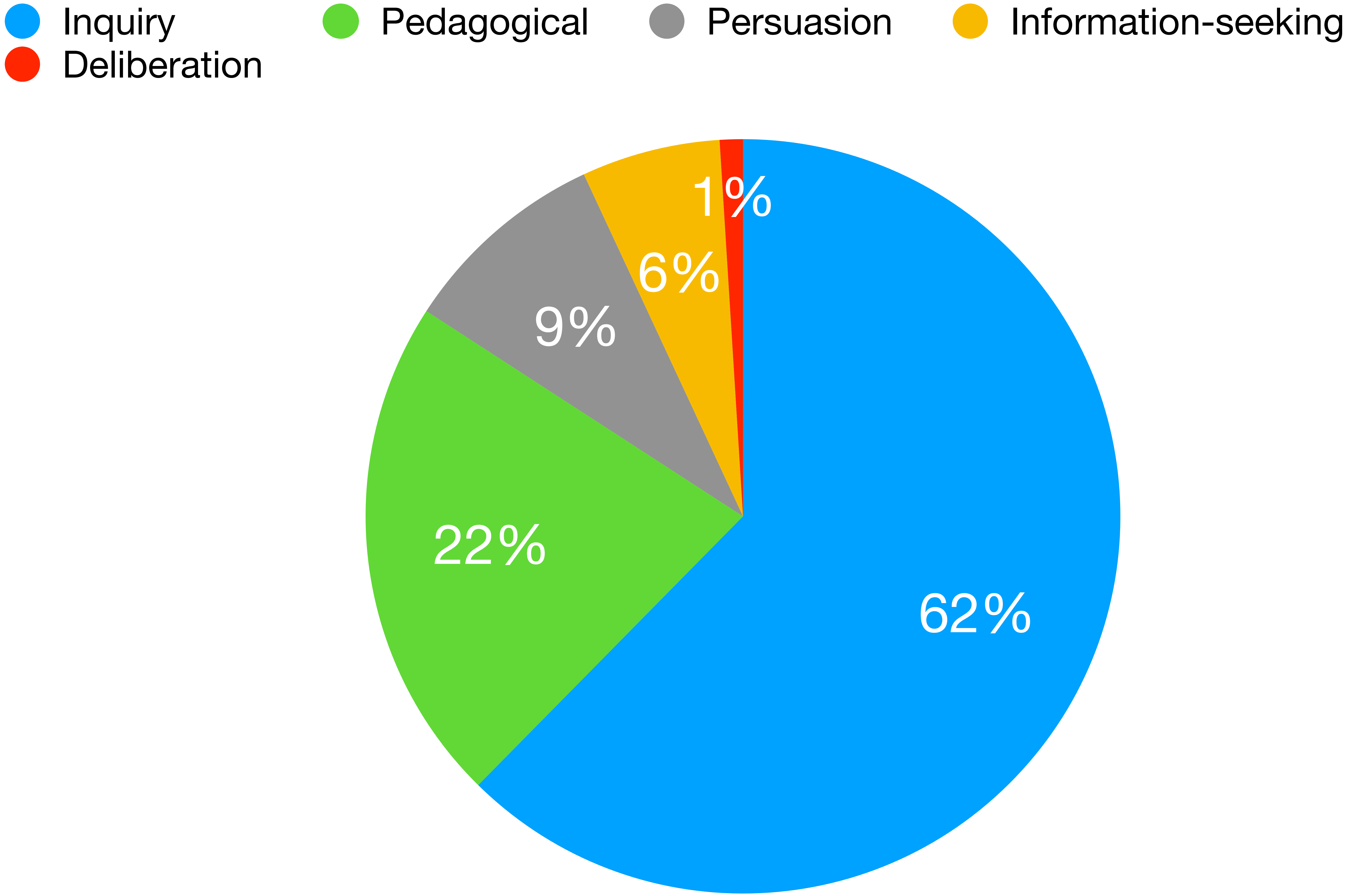


The Mathematical Context:
What kind of mathematical object does the explanation concern?



The Social Context:

In what sort of dialogue does the explanation take place?



Findings

- There is a tradition of explanation in mathematical practice.
- All explanations are answers to why-questions.
- Explanation does not occur primarily as an appeal to a higher level of generality.
- Explanations can be categorised as either trace explanations, strategic explanations, or deep explanations.
- Explanations in mathematics contain purposive elements.
- Explanations can occur in many mathematical contexts.
- Explanations in mathematics are a social phenomenon.

Lessons for ATP

- Explanation is very common in mathematical discourse (accounting for nearly 2/5's of the conversation).
- Explanations in maths:
  - take into account who is explaining and to whom
  - occur in many mathematical contexts and concern many types of mathematical object
  - occur throughout a proof attempt (not just at the end)

Conclusions

1: Look at the backstage

- Computer support for mathematics, such as computer algebra or computational mathematics, has typically been for the frontstage. A second approach is to focus on the backstage and to try to extract principles which are sufficiently clear as to allow an algorithmic interpretation.
- The mechanisms by which research mathematics progresses -- as messy, fallible, and speculative as this may be -- can usefully be studied via analysis of informal mathematics.
- The study of mathematical practice, via philosophy and sociology of mathematical practice provides an excellent starting point for this work.

Conclusions
2: Consider the wider context

- New mathematical knowledge is more than new mathematical proofs. Conjecture and concept generation are subject to rationality as well as proof, and therefore systems can be developed which integrate these theory-development aspects alongside proof generation.

Conclusions
3: Consider social aspects

- Mathematics is social.
- Extending the power and reach of MathOverflow or Polymath through a combination of people and machines raises new challenges for artificial intelligence and computational mathematics.

Conclusions

3: Consider social aspects

- Likely mathematical elements of a mathematics social machine would include the following:
  - Databases of examples, perhaps incorporating user tagging, and also of being able to mine libraries for data and deductions beyond the immediate facts they record.
  - Technology for going beyond the linear structure to capture the more complex structure of a proof attempt, or to represent diagrams.
  - Automated theory formation systems which automatically invent concepts and conjectures.
  - The importance of collaborative systems that “think like a mathematician”

References

How do mathematicians communicate?

What do mathematicians talk about?

How do mathematicians explain things?

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