Towards computer-assisted proofs of parametric Andrews-Curtis simplifications

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Andrews-Curtis Conjecture. Preliminaries

For a group presentation $\langle x_1, \ldots, x_n; r_1, \ldots r_m \rangle$ with generators x_i, and relators r_j, consider the following transformations.

- AC1 Replace some r_i by r_i^{-1}.
- AC2 Replace some r_i by $r_i \cdot r_j$, $j \neq i$.
- AC3 Replace some r_i by $w \cdot r_i \cdot w^{-1}$ where w is any word in the generators.
- AC4 Introduce a new generator y and relator y or delete a generator y and relator y.

Andrews-Curtis Conjecture

- Two presentations g and g' are called *Andrews-Curtis equivalent (AC-equivalent)* if one of them can be obtained from the other by applying a finite sequence of transformations of the types (AC1) - (AC3). Two presentations are *stably AC-equivalent* if one of them can be obtained from the other by applying a finite sequence of transformations of the types (AC1) - (AC4).
- A group presentation $g = \langle x_1, \ldots, x_n; r_1, \ldots r_m \rangle$ is called *balanced* if $n = m$, that is a number of generators is the same as a number of relators. Such n we call a *dimension* of g and denote by $Dim(g)$.

Conjecture (1965)

if $\langle x_1, \ldots, x_n; r_1, \ldots r_n \rangle$ is a balanced presentation of the trivial group it is (stably) AC-equivalent to the trivial presentation $\langle x_1, \ldots, x_n; x_1, \ldots x_n \rangle$.

Trivial Example

- $\langle a, b \mid ab, b \rangle \to \langle a, b \mid ab, b^{-1} \rangle \to \langle a, b \mid a, b^{-1} \rangle \to \langle a, b \mid a, b \rangle$

AC-conjecture: short profile

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- How to find simplifications, algorithmically?
- If a simplification exists, it could be found by the exhaustive search/total enumeration (iterative deepening)
- The issue: simplifications could be very long (Bridson 2015; Lishak 2015)

Search of trivializations and elimination of counterexamples

- Genetic search algorithms (Miasnikov 1999; Swan et al. 2012)
- Breadth-First search (Havas-Ramsay, 2003; McCaul-Bowman, 2006)
- Todd-Coxeter coset enumeration algorithm (Havas-Ramsay,2001)
- Generalized moves and strong equivalence relations (Panteleev-Ushakov, 2016)
- ...

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Our approach: apply generic automated reasoning instead of specialized algorithms (L,2018-..)
Our Claim: generic automated reasoning is (very) competitive

ACT rewriting system, dim =2

Equational theory of groups T_G:

- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- $x \cdot e = x$
- $e \cdot x = x$
- $x \cdot r(x) = e$

For each $n \geq 2$ we formulate a term rewriting system modulo T_G, which captures AC-transformations of presentations of dimension n.
For an alphabet $A = \{a_1, a_2\}$ a term rewriting system ACT_2 consists the following rules:

$$
\begin{array}{rl}
\text{R1L} & f(x,y) \to f(r(x),y)) \\
\text{R1R} & f(x,y) \to f(x,r(y)) \\
\text{R2L} & f(x,y) \to f(x \cdot y, y) \\
\text{R2R} & f(x,y) \to f(x, y \cdot x) \\
\text{R3L}_i & f(x,y) \to f((a_i \cdot x) \cdot r(a_i), y) \text{ for } a_i \in A, i = 1,2 \\
\text{R3R}_i & f(x,y) \to f(x, (a_i \cdot y) \cdot r(a_i)) \text{ for } a_i \in A, i = 1,2
\end{array}
$$

AC-transformations as rewriting modulo group theory

The rewrite relation $\rightarrow_{ACT/G}$ for ACT modulo theory T_G:
$t \rightarrow_{ACT/G} s$ iff there exist $t' \in [t]_G$ and $s' \in [s]_G$ such that $t' \rightarrow_{ACT} s'$.

Reduced ACT_2

Reduced term rewriting system $rACT_2$ consists of the following rules:

R1L $f(x,y) \to f(r(x),y))$

R2L $f(x,y) \to f(x \cdot y, y)$

R2R $f(x,y) \to f(x, y \cdot x)$

R3L$_i$ $f(x,y) \to f((a_i \cdot x) \cdot r(a_i), y)$ for $a_i \in A, i = 1,2$

Proposition

Term rewriting systems ACT_2 and $rACT_2$ considered modulo T_G are equivalent, that is $\to^*_{ACT_2/G}$ and $\to^*_{rACT_2/G}$ coincide.

Proposition

For ground t_1 and t_2 we have $t_1 \to^*_{ACT_2/G} t_2 \Leftrightarrow t_2 \to^*_{ACT_2/G} t_1$, that is $\to^*_{ACT_2/G}$ is symmetric.

Equational Translation

Denote by E_{ACT_2} an equational theory $T_G \cup rACT^{=}$ where $rACT^{=}$ includes the following axioms (equality variants of the above rewriting rules):

E-R1L $f(x, y) = f(r(x), y))$

E-R2L $f(x, y) = f(x \cdot y, y)$

E-R2R $f(x, y) = f(x, y \cdot x)$

E-R3L$_i$ $f(x, y) = f((a_i \cdot x) \cdot r(a_i), y)$ for $a_i \in A, i = 1, 2$

Proposition

For ground terms t_1 and t_2 $t_1 \rightarrow^*_{ACT_2/G} t_2$ iff $E_{ACT_2} \vdash t_1 = t_2$

A variant of the equational translation: replace the axioms $\mathbf{E - R3L_i}$ by "non-ground" axiom $\mathbf{E - RLZ}$: $f(x, y) = f((z \cdot x) \cdot r(z), y)$

Implicational Translation

Denote by I_{ACT_2} the first-order theory $T_G \cup rACT_2^{\rightarrow}$ where $rACT_2^{\rightarrow}$ includes the following axioms:

I-R1L $R(f(x,y)) \rightarrow R(f(r(x),y)))$

I-R2L $R(f(x,y)) \rightarrow R(f(x \cdot y, y))$

I-R2R $R(f(x,y)) \rightarrow R(f(x, y \cdot x))$

I-R3L$_i$ $R(f(x,y)) \rightarrow R(f((a_i \cdot x) \cdot r(a_i), y))$ for $a_i \in A, i = 1,2$

Proposition

For ground terms t_1 and t_2 $t_1 \rightarrow^*_{ACT_2/G} t_2$ iff $I_{ACT_2} \vdash R(t_1) \rightarrow R(t_2)$

Automated Reasoning for AC conjecture exploration

For any pair of presentations p_1 and p_2,
to establish whether they are AC-equivalent one can formulate and try to solve first-order theorem proving problems

- $E_{ACT_n} \vdash t_{p_1} = t_{p_2}$, or
- $I_{ACT_n} \vdash R(t_{p_1}) \rightarrow R(t_{p_2})$

OR, theorem disproving problems

- $E_{ACT_n} \not\vdash t_{p_1} = t_{p_2}$, or
- $I_{ACT_n} \not\vdash R(t_{p_1}) \rightarrow R(t_{p_2})$

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Our proposal: apply automated reasoning: ATP and finite model building.

Theorem Proving for AC-Simplifications

Elimination of potential counterexamples

- **Known cases:** We have applied automated theorem proving using Prover9 prover (McCune, 2007) to confirm that all cases eliminated as potential counterexamples in all known literature can be eliminated by our method too.

Theorem Proving for AC-Simplifications (cont.)

New cases (from Edjvet-Swan, 2005-2010):

T14 $\langle a, b \mid ababABB, babaBAA \rangle$
T28 $\langle a, b \mid aabbbbABBBB, bbaaaaBAAAA \rangle$
T36 $\langle a, b \mid aababAABB, bbabaBBAA \rangle$
T62 $\langle a, b \mid aaabbAbABBB, bbbaaBaBAAA \rangle$
T74 $\langle a, b \mid aabaabAAABB, bbabbaBBBAA \rangle$

T16 $\langle a, b, c \mid ABCacbb, BCAbacc, CABcbaa \rangle$
T21 $\langle a, b, c \mid ABCabac, BCAbcba, CABcacb \rangle$
T48 $\langle a, b, c \mid aacbcABCC, bbacaBCAA, ccbabCABB \rangle$
T88 $\langle a, b, c \mid aacbAbCAB, bbacBcABC, ccbaCaBCA \rangle$
T89 $\langle a, b, c \mid aacbcACAB, bbacBABC, ccbaCBCA \rangle$

T96 $\langle a, b, c, d \mid adCADbc, baDBAcd, cbACBda, dcBDCab \rangle$
T97 $\langle a, b, c, d \mid adCAbDc, baDBcAd, cbACdBa, dcBDaCb \rangle$ [ICMS 2018]

Miller-Shupp presentations

- $MS_n(w) = \langle x, y \mid x^{-1}y^n x = y^{n+1}, x = w \rangle$ where w is a word in x and y with exponent sum 0 on x, and $n > 0$ is a balanced presentation of trivial group (Miller-Shupp, 1999)
- $MS_n(w_*)$ is well-known subfamily with $w_* = y^{-1}xyx^{-1}$
- $MS_n(w_*)$ is AC-trivializable for $n \leq 2$ (Miasnikov 1999; Havas-Ramsay, 2003)
- $MS_3(w_*)$ is *stably* AC-trivializable (Fernández, 2019)

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- We show: $MS_n(w_*)$ **is AC-trivializable for n=3,4,5,6,7** using automated theorem proving (new).

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- We show: $MS_n(w_*)$ **is AC-trivializable for n=3,4,5,6,7** using automated theorem proving (new).
- **Ultimately we would like to get an inductive proof for all $n \geq 2$ by generalization of automated proofs. We are not there yet.**

Something simpler: pseudo Miller-Shupp presentations

- $pMS_n(w) = \langle x, y \mid x^{-1}y^n x = y^{n+1}, x^{-1} = w \rangle$ where w is a word in x and y with exponent sum 0 on x, and $n > 0$ is a balanced presentation of trivial group
- $pMS_n(w_*)$ is a subfamily with $w_* = y^{-1}xyx^{-1}$
- We show: $pMS_n(w_*)$ **is AC-trivializable for all** $n > 2$ using AR-assisted proof with implicational encoding.

Proof illustration, I

Steps	Lines	n=3	n=4	n=5
1	40	$\langle a^{-1}b^3ab^{-4}, ab^{-1}aba^{-1}\rangle$	$\langle a^{-1}b^4ab^{-5}, ab^{-1}aba^{-1}\rangle$	$\langle a^{-1}b^5ab^{-6}, ab^{-1}aba^{-1}\rangle$
2	45	$\langle b^3ab^{-4}a^{-1}, ab^{-1}aba^{-1}\rangle$	$\langle b^4ab^{-5}a^{-1}, ab^{-1}aba^{-1}\rangle$	$\langle b^5ab^{-6}a^{-1}, ab^{-1}aba^{-1}\rangle$
3	49	$\langle b^3ab^{-4}a^{-1}, ab^{-1}a^{-1}ba^{-1}\rangle$	$\langle b^4ab^{-5}a^{-1}, ab^{-1}a^{-1}aba^{-1}\rangle$	$\langle b^5ab^{-6}a^{-1}, ab^{-1}a^{-1}ba^{-1}\rangle$
4	50	$\langle b^3ab^{-5}a^{-1}ba^{-1}, ab^{-1}a^{-1}ba^{-1}\rangle$	$\langle b^4ab^{-6}a^{-1}ba^{-1}, ab^{-1}a^{-1}aba^{-1}\rangle$	$\langle b^5ab^{-7}a^{-1}ba^{-1}, ab^{-1}a^{-1}ba^{-1}\rangle$
5	51	$\langle ab^{-1}ab^5a^{-1}b^{-3}, ab^{-1}a^{-1}ba^{-1}\rangle$	$\langle ab^{-1}ab^6a^{-1}b^{-4}, ab^{-1}a^{-1}aba^{-1}\rangle$	$\langle ab^{-1}ab^7a^{-1}b^{-5}, ab^{-1}a^{-1}ba^{-1}\rangle$
6	52	$\langle ab^{-1}ab^5a^{-1}b^{-3}, ab^4a^{-1}b^{-3}\rangle$	$\langle ab^{-1}ab^6a^{-1}b^{-4}, ab^5a^{-1}b^{-4}\rangle$	$\langle ab^{-1}ab^7a^{-1}b^{-5}, ab^6a^{-1}b^{-5}\rangle$
7	53	$\langle ab^{-1}ab^5a^{-1}b^{-3}, b^3ab^{-4}a^{-1}\rangle$	$\langle ab^{-1}ab^6a^{-1}b^{-4}, b^4ab^{-5}a^{-1}\rangle$	$\langle ab^{-1}ab^7a^{-1}b^{-5}, b^5ab^{-6}a^{-1}\rangle$
8	54	$\langle ab^{-1}aba^{-1}, b^3ab^{-4}a^{-1}\rangle$	$\langle ab^{-1}aba^{-1}, b^4ab^{-5}a^{-1}\rangle$	$\langle ab^{-1}aba^{-1}, b^5ab^{-6}a^{-1}\rangle$
9	55	$\langle bab^{-1}aba^{-1}b^{-1}, b^3ab^{-4}a^{-1}\rangle$	$\langle bab^{-1}aba^{-1}b^{-1}, b^4ab^{-5}a^{-1}\rangle$	$\langle bab^{-1}aba^{-1}b^{-1}, b^5ab^{-6}a^{-1}\rangle$
10	56	$\langle b^2ab^{-1}aba^{-1}b^{-2}, b^3ab^{-4}a^{-1}\rangle$	$\langle b^2ab^{-1}aba^{-1}b^{-2}, b^4ab^{-5}a^{-1}\rangle$	$\langle b^2ab^{-1}aba^{-1}b^{-2}, b^5ab^{-6}a^{-1}\rangle$
11	57	$\langle b^3ab^{-1}aba^{-1}b^{-3}, b^3ab^{-4}a^{-1}\rangle$	$\langle b^3ab^{-1}aba^{-1}b^{-3}, b^4ab^{-5}a^{-1}\rangle$	$\langle b^3ab^{-1}aba^{-1}b^{-3}, b^5ab^{-6}a^{-1}\rangle$
12	58	$\langle b^3ab^{-1}ab^{-3}a^{-1}, b^3ab^{-4}a^{-1}\rangle$	$\langle b^4ab^{-1}aba^{-1}b^{-4}, b^4ab^{-5}a^{-1}\rangle$	$\langle b^4ab^{-1}aba^{-1}b^{-4}, b^5ab^{-6}a^{-1}\rangle$
13	59	$\langle b^3ab^{-1}ab^{-3}a^{-1}, ab^4a^{-1}b^{-3}\rangle$	$\langle b^4ab^{-1}ab^{-4}a^{-1}, b^4ab^{-5}a^{-1}\rangle$	$\langle b^5ab^{-1}aba^{-1}b^{-5}, b^5ab^{-6}a^{-1}\rangle$
14	60	$\langle b^3ab^{-1}ab^{-3}a^{-1}, ab^3ab^{-3}a^{-1}\rangle$	$\langle b^4ab^{-1}ab^{-4}a^{-1}, ab^5a^{-1}b^{-4}\rangle$	$\langle b^5ab^{-1}ab^{-5}a^{-1}, b^5ab^{-6}a^{-1}\rangle$
15	61	$\langle b^3ab^{-1}ab^{-3}a^{-1}, ab^3a^{-1}b^{-3}a^{-1}\rangle$	$\langle b^4ab^{-1}ab^{-4}a^{-1}, ab^4ab^{-4}a^{-1}\rangle$	$\langle b^5ab^{-1}ab^{-5}a^{-1}, ab^6a^{-1}b^{-5}\rangle$
16	62	$\langle b^3ab^{-4}a^{-1}, ab^3a^{-1}b^{-3}a^{-1}\rangle$	$\langle b^4ab^{-1}ab^{-4}a^{-1}, ab^4a^{-1}b^{-4}a^{-1}\rangle$	$\langle b^5ab^{-1}ab^{-5}a^{-1}, ab^5ab^{-5}a^{-1}\rangle$
17	63	$\langle ab^4a^{-1}b^{-3}, ab^3a^{-1}b^{-3}a^{-1}\rangle$	$\langle b^4ab^{-5}a^{-1}, ab^4a^{-1}b^{-4}a^{-1}\rangle$	$\langle b^5ab^{-1}ab^{-5}a^{-1}, ab^5a^{-1}b^{-5}a^{-1}\rangle$
18	64	$\langle ab^4a^{-1}b^{-3}, ab^3ab^{-3}a^{-1}\rangle$	$\langle ab^5a^{-1}b^{-4}, ab^4a^{-1}b^{-4}a^{-1}\rangle$	$\langle b^5ab^{-6}ab^{-5}a^{-1}, ab^5a^{-1}b^{-5}a^{-1}\rangle$
19	65	$\langle a^2b^4a^{-1}b^{-3}a^{-1}, ab^3ab^{-3}a^{-1}\rangle$	$\langle ab^5a^{-1}b^{-4}, ab^4ab^{-4}a^{-1}\rangle$	$\langle ab^6a^{-1}b^{-5}, ab^5a^{-1}b^{-5}a^{-1}\rangle$
20	66	$\langle a^2ba^{-1}, ab^3ab^{-3}a^{-1}\rangle$	$\langle a^2b^5a^{-1}b^{-4}a^{-1}, ab^4ab^{-4}a^{-1}\rangle$	$\langle ab^6a^{-1}b^{-5}, ab^5ab^{-5}a^{-1}\rangle$
21	67	$\langle a^2ba^{-1}, ab^3a^{-1}b^{-3}a^{-1}\rangle$	$\langle a^2ba^{-1}, ab^4ab^{-4}a^{-1}\rangle$	$\langle a^2b^6a^{-1}b^{-5}a^{-1}, ab^5ab^{-5}a^{-1}\rangle$
22	68	$\langle ab^{-1}a^{-2}, ab^3a^{-1}b^{-3}a^{-1}\rangle$	$\langle a^2ba^{-1}, ab^4a^{-1}b^{-4}a^{-1}\rangle$	$\langle a^2ba^{-1}, ab^5ab^{-5}a^{-1}\rangle$
23	69	$\langle ab^{-1}a^{-2}, ab^3a^{-1}b^{-4}a^{-2}\rangle$	$\langle ab^{-1}a^{-2}, ab^4a^{-1}b^{-4}a^{-1}\rangle$	$\langle a^2ba^{-1}, ab^5a^{-1}b^{-5}a^{-1}\rangle$
24	70	$\langle ab^{-1}a^{-2}, a^2b^4ab^{-3}a^{-1}\rangle$	$\langle ab^{-1}a^{-2}, ab^4a^{-1}b^{-5}a^{-2}\rangle$	$\langle ab^{-1}a^{-2}, ab^5a^{-1}b^{-5}a^{-1}\rangle$
25	71	$\langle ab^3ab^{-3}a^{-1}, a^2b^4ab^{-3}a^{-1}\rangle$	$\langle ab^{-1}a^{-2}, a^2b^5ab^{-4}a^{-1}\rangle$	$\langle ab^{-1}a^{-2}, ab^5a^{-1}b^{-6}a^{-2}\rangle$
26	72	$\langle ab^3a^{-1}b^{-3}a^{-1}, a^2b^4ab^{-3}a^{-1}\rangle$	$\langle ab^4ab^{-4}a^{-1}, a^2b^5ab^{-4}a^{-1}\rangle$	$\langle ab^{-1}a^{-2}, a^2b^6ab^{-5}a^{-1}\rangle$
27	73	$\langle \mathbf{ab^3a^{-1}b^{-3}a^{-1}, a^2ba^{-1}}\rangle$	$\langle ab^4a^{-1}b^{-4}a^{-1}, a^2b^5ab^{-4}a^{-1}\rangle$	$\langle ab^5ab^{-5}a^{-1}, a^2b^6ab^{-5}a^{-1}\rangle$
28	74	$\langle ab^3ab^{-3}a^{-1}, a^2ba^{-1}\rangle$	$\langle ab^4a^{-1}b^{-4}a^{-1}, a^2ba^{-1}\rangle$	$\langle ab^5a^{-1}b^{-5}a^{-1}, a^2b^6ab^{-5}a^{-1}\rangle$
29	75	$\langle a^2b^3ab^{-3}a^{-2}, a^2ba^{-1}\rangle$	$\langle ab^4ab^{-4}a^{-1}, a^2ba^{-1}\rangle$	$\langle ab^5a^{-1}b^{-5}a^{-1}, a^2ba^{-1}\rangle$
30	76	$\langle a^2b^3ab^{-2}a^{-1}, a^2ba^{-1}\rangle$	$\langle a^2b^4ab^{-4}a^{-2}, a^2ba^{-1}\rangle$	$\langle ab^5ab^{-5}a^{-1}, a^2ba^{-1}\rangle$
31	77	$\langle ab^2a^{-1}b^{-3}a^{-2}, a^2ba^{-1}\rangle$	$\langle a^2b^4ab^{-3}a^{-1}, a^2ba^{-1}\rangle$	$\langle a^2b^5ab^{-5}a^{-2}, a^2ba^{-1}\rangle$
32	78	$\langle ab^2a^{-1}b^{-2}a^{-1}, a^2ba^{-1}\rangle$	$\langle ab^3a^{-1}b^{-4}a^{-2}, a^2ba^{-1}\rangle$	$\langle a^2b^5ab^{-4}a^{-1}, a^2ba^{-1}\rangle$
33	79	$\langle ab^2ab^{-2}a^{-1}, a^2ba^{-1}\rangle$	$\langle \mathbf{ab^3a^{-1}b^{-3}a^{-1}, a^2ba^{-1}}\rangle$	$\langle ab^4a^{-1}b^{-5}a^{-2}, a^2ba^{-1}\rangle$
34	80	$\langle a^2b^2ab^{-2}a^{-2}, a^2ba^{-1}\rangle$	$\langle ab^3ab^{-3}a^{-1}, a^2ba^{-1}\rangle$	$\langle ab^4a^{-1}b^{-4}a^{-1}, a^2ba^{-1}\rangle$
35	81	$\langle a^2b^2ab^{-1}a^{-1}, a^2ba^{-1}\rangle$	$\langle a^2b^3ab^{-3}a^{-2}, a^2ba^{-1}\rangle$	$\langle ab^4ab^{-4}a^{-1}, a^2ba^{-1}\rangle$
36	82	$\langle aba^{-1}b^{-2}a^{-2}, a^2ba^{-1}\rangle$	$\langle a^2b^3ab^{-2}a^{-1}, a^2ba^{-1}\rangle$	$\langle a^2b^4ab^{-4}a^{-2}, a^2ba^{-1}\rangle$
37	83	$\langle aba^{-1}b^{-1}a^{-1}, a^2ba^{-1}\rangle$	$\langle ab^2a^{-1}b^{-3}a^{-2}, a^2ba^{-1}\rangle$	$\langle a^2b^4ab^{-3}a^{-1}, a^2ba^{-1}\rangle$
38	84	$\langle abab^{-1}a^{-1}, a^2ba^{-1}\rangle$	$\langle ab^2a^{-1}b^{-2}a^{-1}, a^2ba^{-1}\rangle$	$\langle ab^3a^{-1}b^{-4}a^{-2}, a^2ba^{-1}\rangle$
39	85	$\langle a^2bab^{-1}a^{-2}, a^2ba^{-1}\rangle$	$\langle ab^2ab^{-2}a^{-1}, a^2ba^{-1}\rangle$	$\langle \mathbf{ab^3a^{-1}b^{-3}a^{-1}, a^2ba^{-1}}\rangle$

Proof illustration, II

Steps	Lines	n=3	n=4	n=5
40	86	$\langle a^2b, a^2ba^{-1}\rangle$	$\langle a^2b^2ab^{-2}a^{-2}, a^2ba^{-1}\rangle$	$\langle ab^3ab^{-3}a^{-1}, a^2ba^{-1}\rangle$
41	87	$\langle a^2b, ab^{-1}a^{-2}\rangle$	$\langle a^2b^2ab^{-1}a^{-1}, a^2ba^{-1}\rangle$	$\langle a^2b^3ab^{-3}a^{-2}, a^2ba^{-1}\rangle$
42	88	$\langle a^2b, a\rangle$	$\langle aba^{-1}b^{-2}a^{-2}, a^2ba^{-1}\rangle$	$\langle a^2b^3ab^{-2}a^{-1}, a^2ba^{-1}\rangle$
43	89	$\langle ba^2, a\rangle$	$\langle aba^{-1}b^{-1}a^{-1}, a^2ba^{-1}\rangle$	$\langle ab^2a^{-1}b^{-3}a^{-2}, a^2ba^{-1}\rangle$
44	90	$\langle ba^2, a^{-1}\rangle$	$\langle abab^{-1}a^{-1}, a^2ba^{-1}\rangle$	$\langle ab^2a^{-1}b^{-2}a^{-1}, a^2ba^{-1}\rangle$
45	91	$\langle ba, a^{-1}\rangle$	$\langle a^2bab^{-1}a^{-2}, a^2ba^{-1}\rangle$	$\langle ab^2ab^{-2}a^{-1}, a^2ba^{-1}\rangle$
46	92	$\langle ab, a^{-1}\rangle$	$\langle a^2b, a^2ba^{-1}\rangle$	$\langle a^2b^2ab^{-2}a^{-2}, a^2ba^{-1}\rangle$
47	93	$\langle ab, b\rangle$	$\langle a^2b, ab^{-1}a^{-2}\rangle$	$\langle a^2b^2ab^{-1}a^{-1}, a^2ba^{-1}\rangle$
48	94	$\langle ab, b^{-1}\rangle$	$\langle a^2b, a\rangle$	$\langle aba^{-1}b^{-2}a^{-2}, a^2ba^{-1}\rangle$
49	95	$\langle a, b^{-1}\rangle$	$\langle ba^2, a\rangle$	$\langle aba^{-1}b^{-1}a^{-1}, a^2ba^{-1}\rangle$
50	96	$\langle a, b\rangle$	$\langle ba^2, a^{-1}\rangle$	$\langle abab^{-1}a^{-1}, a^2ba^{-1}\rangle$
51	97		$\langle ba, a^{-1}\rangle$	$\langle a^2bab^{-1}a^{-2}, a^2ba^{-1}\rangle$
52	98		$\langle ab, a^{-1}\rangle$	$\langle a^2b, a^2ba^{-1}\rangle$
53	99		$\langle ab, b\rangle$	$\langle a^2b, ab^{-1}a^{-2}\rangle$
54	100		$\langle ab, b^{-1}\rangle$	$\langle a^2b, a\rangle$
55	101		$\langle a, b^{-1}\rangle$	$\langle ba^2, a\rangle$
56	102		$\langle a, b\rangle$	$\langle ba^2, a^{-1}\rangle$
57	103			$\langle ba, a^{-1}\rangle$
58	104			$\langle ab, a^{-1}\rangle$
59	105			$\langle ab, b\rangle$
60	106			$\langle ab, b^{-1}\rangle$
61	107			$\langle a, b^{-1}\rangle$
62	108			$\langle a, b\rangle$

Table 3. Configurations/presentations reached in proofs at steps 40–62 (n=3,4,5)

Back to real (not pseudo) $MS_n(w_*)$ presentations

Not so simple!

Implicational encoding of $MS_n(w_*)$ for $2 \leq n \leq 6$

n	simplification steps	time, s
2	34	0.05
3	85	0.66
4	242	5.97
5	573	265
6	1282	10637

Equational encoding of $MS_7(w_*)$

n	simplification macrosteps	time, s
7	892	42681

- Equational proof uses multiple lemmas, each corresponding to a macrostep in AC-simplifications
- Example of a lemma:
  f(x * y,y * (z' * (y * x'))) = f(x * y,x * (x * (x * z))).

Observations and conjectures

Conjecture

All presentations $MS_n(w_*)$ are AC-trivializable for $n \geq 3$ using the following sequence of transformations
$MS_n(w_*) \Rightarrow^* \langle a, b | b^{-(n-1)} a^{-4} ba, w_1 \rangle \Rightarrow^* \ldots \Rightarrow^*$
$\langle a, b | b^{-(n-k)} a^{-4} ba, w_k \rangle \Rightarrow^* \ldots \Rightarrow^* \langle a, b | b^{-2} a^{-4} ba, w_{n-2} \rangle \Rightarrow^* \langle a, b | a, b \rangle$,
$k = 1 \ldots n-2$, *where* $w_k = a^{-1} b^{-1} aba^{-1}$ *or* $w_k = ab^{-1} a^{-1} ba$.

Supported by obtained simplifications for n=3,4,5

Example (n=5):
$\Rightarrow^* \langle a, b | b^{-4} a^{-4} ba, w_1 \rangle \Rightarrow^* \langle a, b | b^{-3} a^{-4} ba, w_2 \rangle \Rightarrow^* \langle a, b | b^{-2} a^{-4} ba, w_3 \rangle$
$\Rightarrow^* \langle a, b | a, b \rangle$

Conclusion

- Automated Proving and Disproving is an interesting and powerful approach to AC-conjecture exploration;
- Source of interesting challenging problems for ATP/ATD;
- Can ML/DM help to guide the proofs and understand the proofs?
- Does AC-conjecture hold true for all $MS_n(w_*)$, n> 0?

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Thank you!