Towards computer-assisted proofs of parametric Andrews-Curtis simplifications

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Abstract

We present recent developments in the applications of automated theorem proving in the investigation of the Andrews-Curtis conjecture. We demonstrate previously unknown simplifications of groups presentations from a parametric family $MS_n(w_*)$ of trivial group presentations for $n = 3, 4, 5, 6$ (subset of well-known Miller-Schupp family). Based on the human analysis of these simplifications we formulate a conjecture on the structure of simplifications for the infinite family $MS_n(w_*)$, $n \geq 3$. We discuss the applications of the proposed methodology to other families of presentations.

Introduction and Outline

The Andrews-Curtis conjecture (ACC) [1] is one of the most well-known open problems in combinatorial group theory. In short, it states that every balanced presentation of the trivial group can be transformed into a trivial presentation by a sequence of simple transformations. Various computational approaches have been proposed for the efficient search of such simplifications, see e.g. [3, 10, 12, 6, 4]. Still there are infinite families of balanced trivial group presentations which remain potential counterexamples to the conjecture, that is for which the required simplifications are not known.

For a group presentation $\langle x_1, \ldots, x_n; r_1, \ldots r_m \rangle$ with generators x_i, and relators r_j, consider the following transformations.

AC1 Replace some r_i by r_i^{-1}.

AC2 Replace some r_i by $r_i \cdot r_j$, $j \neq i$.

AC3 Replace some r_i by $w \cdot r_i \cdot w^{-1}$ where w is any word in the generators.

AC4 Introduce a new generator y and relator y or delete a generator y and relator y.

Two presentations g and g' are called *Andrews-Curtis equivalent (AC-equivalent)* if one of them can be obtained from the other by applying a finite sequence of transformations of the types (AC1) - (AC3). Two presentations are stably AC-equivalent if one of them can be obtained from the other by applying a finite sequence of transformations of the types (AC1)–(AC4). A presentation $\langle x_1, \ldots, x_n; r_1, \ldots r_m \rangle$ is called *balanced* if $n = m$.

Conjecture 1 (Andrews-Curtis [1]). *If $\langle x_1, \ldots, x_n; r_1, \ldots r_n \rangle$ is a balanced presentation of the trivial group it is AC-equivalent to the trivial presentation $\langle x_1, \ldots, x_n; x_1, \ldots x_n \rangle$*

The weak form of the conjecture states that every balanced presentation for a trivial group is stably AC-equivalent (i.e. transformations AC4 are allowed) to the trivial presentation. Both variants of the conjecture remain open and challenging problems.

Miller-Schupp presentations

In [5] the authors have defined an infinite family of balanced presentations of the trivial group $MS_n(w) = \langle a, b \mid a^{-1}b^n a = b^{n+1}, a = w\rangle$, where $n \geq 1$ and w is a word which has exponent sum 0 on a. Since these presentations have been used as a test-bed for testing various computational methods for finding AC-trivializations, see e.g. [3, 10, 11, 2]. Both novel trivializations and some remaining open cases for n=2 can be found in [11]. Subfamily $MS_n(w_*)$ for a fixed $w_* = b^{-1}aba^{-1}$, $n \geq 1$ was considered in [3, 10, 2]. The trivializations for $MS_n(w_*)$, $n \leq 2$ were demonstrated in [3, 10], while in [2] it was shown that $MS_3(w_*)$ is *stably* AC- trivializable. The AC-trivializability of cases of $MS_n(w_*)$ with $n \geq 3$ remained open [2].

Our contribution

In [7, 8, 9] we have developed an approach based on using automated deduction in first-order logic in the search of trivializations and have shown that the approach is very competitive. In our approach we formalized the AC-transformations in terms of term rewriting modulo group theory and first-order deduction. In the research reported in this abstract we demonstrate new AC-trivializations obtained by automated reasoning:

Proposition 1. *Group presentations $MS_n(w_*)$ are AC-trivializable for n=3,4,5,6*

These trivializations were found by automated theorem proving using Prover9 prover. We have published all proofs and extracted trivializations online [1].

Our ongoing work includes analysis of these long sequences of transformations in order to comprehend and generalize these proofs with the aim to arrive at general and likely inductive argument of trivializability applicable to the whole family $MS_n(w_*), n \geq 3$. While we were not able to complete it yet the analysis for n=3,4,5 has shown that the proofs demonstrate some regularity, which we formalize in the following conjecture.

Conjecture 2. *All presentations $MS_n(w_*)$ are AC-trivializable for $n \geq 3$ using the following sequence of transformations*

$MS_n(w_*) \Rightarrow^* \langle a, b | b^{-(n-1)}a^{-4}ba, w_1\rangle \Rightarrow^* \ldots \Rightarrow^* \langle a, b | b^{-(n-k)}a^{-4}ba, w_k\rangle \Rightarrow^* \ldots \Rightarrow^* \langle a, b | b^{-2}a^{-4}ba, w_{n-2}\rangle \Rightarrow^* \langle a, b | a, b\rangle$, *$k = 1 \ldots n-2$, where $w_k = a^{-1}b^{-1}aba^{-1}$ or $w_k = ab^{-1}a^{-1}ba$.*

Interestingly, the only available transformation sequence for n=6 does not fit the pattern indicated in the conjecture. As it is very long sequence (1768 proof steps, obtained in excess of 10,600s) there might well be alternative simplification sequences satisfying the patterns of the conjecture. We tested the methodology "get automated proofs for a few values of parameter, then generalise by human reasoning" for other parametric families of balanced presentations of trivial group. The results are mixed so far. In one case of slightly modified family of $MS_n(w_{**}) = \langle a, b \mid a^{-1}b^n a = b^{n+1}, a^{-1} = w\rangle\}, n \geq 2$ we were able to get an inductive argument for general case by analysis of automated proofs for particular values of n (=2,3,4), but it should not be overestimated as in this case there a simple direct (and different) argument of trivializability, which we leave to an interested reader to find as an exercise. We have shown that generic automated first-order proving can be used in combinatorial group theory, both in tackling open questions and as a competitive alternative to specialized algorithms. Considering parametric families of balanced group presentations brings interesting challenges for automated proofs comprehension, generalisation and regularisation, which could be tackled by combinations of methods from automated reasoning, machine learning, data and process mining. This is subject of our ongoing work.

[1] https://doi.org/10.5281/zenodo.8267429

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