Computer-assisted identification of splittings in subvariety lattices *

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Abstract

We describe a heuristic approach to identifying splittings in some lattices of subvarieties using computer-assisted methods, in particular McCune's PROVER9/MACE4. This approach shows promise to facilitate the analysis of subvariety lattices for many classes of algebraic structures, in particular in situations where the large size of the algebras involved in the splittings make human-executed proofs infeasible.

Equational reasoning can often be fruitfully analyzed from a semantic perspective by considering *varieties*—i.e., classes of algebraic structures satisfying some set of equations, or equivalently that are closed under taking subalgebras, direct products, and homomorphic images. Indeed, given an equational theory $\mathcal{E}$ and the the variety $\mathcal{V}$ of algebras modeling $\mathcal{E}$, there is a bijective correspondence between equational extensions of $\mathcal{E}$ and varieties contained in $\mathcal{V}$. Ordered under inclusion, the varieties contained in $\mathcal{V}$ form a complete lattice. This subvariety lattice of $\mathcal{V}$ completely encodes equational inference in the presence of $\mathcal{E}$ due to the previously mentioned correspondence, and thus understanding the structure of the subvariety lattice is crucial. Because a complete description of the subvariety lattice of a given variety $\mathcal{V}$ is usually not possible, its structure is often illuminated in terms of its *splittings*. These consists of pairs $(\mathcal{W}_1, \mathcal{W}_2)$ of subvarieties of $\mathcal{V}$ such that for each subvariety $\mathcal{W}$ of $\mathcal{V}$, $\mathcal{W} \subseteq \mathcal{W}_1$ if and only if $\mathcal{W}_2 \not\subseteq \mathcal{W}$. The splittings of a subvariety lattice provide ways of partitioning it, and have been studied extensively both in general (see, e.g., [1]) and for certain important varieties of algebras (see, e.g., [4]).

This work presents a case study in identifying such splittings using a computer-assisted approach. Our case study illustrates a heuristic method of finding splittings in suitably-chosen subvariety lattices, and relies on a human-guided computer search. This approach facilitates the understanding of subvariety lattices in situations where human-executed approaches are rendered infeasible by the size of the structures involved, necessitating the use of computational resources. In particular, we execute our heuristic approach using McCune's PROVER9/MACE4 [6] to identify some important splittings in the subvariety lattice of *involutive lattices*. The latter comprise a class of lattice-ordered algebraic structures that contain, *inter alia*, ortholattices and Boolean algebras. We focus on the Kleene identity

$$x \wedge \neg x \leq y \vee \neg y, \tag{K}$$

which plays an extremely important role in the theory of distributive involutive lattices (see, e.g., [5, 7, 2, 3]). We characterize the failure of (K) in arbitrary (not necessarily distributive) involutive lattices by the presence of six forbidden configurations $\mathbf{F}_i$, $i \in \{4, 5, 6, 8, 10, 12\}$.

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These forbidden configurations are found by guided use of Mace4 to construct countermodels witnessing each manner in which (K) may fail. First, Mace4 is asked to produce small countermodels witnessing the failure of (K). After appropriate candidates for forbidden configurations are identified, we scrutinize these for salient features (such as the presence of an involution fixed point, distributivity, or special equations satisfied by generators). This yields a set of conditions $\Sigma_{\mathbf{F}}$, which we conjecture guarantees the presence of the candidate forbidden configuration $\mathbf{F}$ as a subalgebra in any involutive lattice $\mathbf{L}$ refuting (K) and satisfying $\Sigma_{\mathbf{F}}$. We then query Mace4 for countermodels of (K) that refute some conditions in $\Sigma_{\mathbf{F}}$. This process is then iterated.

Although there is no reason *a priori* why this process must terminate in general, for involutive lattices we obtain six involutive lattices $\mathbf{F}_i$, $i \in \{4,5,6,8,10,12\}$, together with six jointly-exhaustive sets of conditions Σ_i, $i \in \{4,5,6,8,10,12\}$ (where i identifies the cardinality of the involutive lattice $\mathbf{F}_i$). In order to prove that these forbidden configurations suffice to characterize the failure of (K), for each i we examine countermodels $\mathbf{L}$ of (K) containing $\mathbf{F}_i$ in order to understand generators of $\mathbf{F}_i$ in $\mathbf{L}$. Specifically, given an involutive lattice $\mathbf{L}$ containing $\mathbf{F}_i$ and a pair of elements $a, b \in L$ witnessing the failure of (K), we identify term functions in the language of involutive lattices that, when given a, b as inputs, produce generators for $\mathbf{F}_i$ as a subalgebra of $\mathbf{L}$.

Once candidate terms of the above kind are identified, we use Prover9 to derive machine proofs that the subalgebras generated by these terms are isomorphic to the appropriate forbidden configuration $\mathbf{F}_i$. It follows from [5] that the involutive lattice $\mathbf{F}_4$ generates the variety of distributive involutive lattices. We further show that for each $i \in \{5,6,8,10,12\}$, the variety generated by $\mathbf{F}_i$ contains $\mathbf{F}_4$. This produces the following splitting result.

Theorem 1. *Let $\mathcal{V}$ be a variety of involutive lattices. Then one of the following holds.*

1. *$\mathcal{V}$ is contained in the variety of involutive lattices satisfying (K).*
2. *$\mathcal{V}$ contains all distributive involutive lattices.*

We expect that the heuristic approach outlined here will be successful in the study of many subvariety lattices, in particular for varieties consisting of lattice-ordered algebraic structures. In addition to discussing these paths forward, we discuss some limitations of this heuristic approach, as well as potential avenues for further automation. In particular, we discuss the possibility of using techniques from machine learning to automate the production of the conditions $\Sigma_{\mathbf{F}}$, as well as the term functions that produce generators of forbidden subalgebras.

References

[1] Day, A.: Splitting algebras and a weak notion of projectivity. Algebra Universalis. 5:153–162, 1975.

[2] Davey, B.A. and Werner, H.: Dualities and equivalences for varieties of algebras. In A.P. Huhn and E.T. Schmidt, editors, *Contributions to Lattice Theory*, pages 101–275. North-Holland, Amsterdam, New York, 1983.

[3] Fussner, W. and Galatos, N.: Categories of models of R-mingle. Ann. Pure Appl. Logic 170(10):1188–1242, 2019.

[4] Jipsen, P. and Rose, H.: *Varieties of Lattices*. Springer-Verlag, Berlin, Heidelberg, 1992.

[5] Kalman, J.: Lattices with involution. Trans. Amer. Math. Soc. 87(2):485–491, 1958.

[6] McCune, W.: *Prover9 and Mace4*. Version Dec-2007. `http://www.cs.unm.edu/~mccune/prover9`

[7] Pynko, A.: Implicational classes of De Morgan lattices. Discrete Mathematics 205:171–181, 1999.