Project Proposal: Neural Modelling of Mathematical Structures

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Section 1

Introduction

Short introduction

Goals

- Build "intuition" for a computer based on models
- Build models of theories based on their axioms
- Try to extend these models
- Guess truthfulness of theorems based on these models (future)

Groups

Group is a structure with functions "composition" ($\cdot$, binary) "inverse" ($^{-1}$, unary) and a constant "unit" (e) that satisfy:

1. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity)
2. $a \cdot e = e \cdot a = a$
3. $a \cdot a^{-1} = a^{-1} \cdot a = e$

Used groups

Cyclic groups

$(\mathbb{Z}_n, +, -, 0)$: Addition modulo n

Permutation groups

$(S_n, \circ, ^{-1}, id)$: Permutations with classic composition, inverse and identity

Section 2

Implementation

Implementation

Elements

Elements are embedded into $\mathbb{R}^n$ with handpicked representations

Functions

Functions are 4-layer feedforward NN, that inputs a vector of size $n \times arity$ and outputs a vector of size n. They are learned by either lookup table or by properties

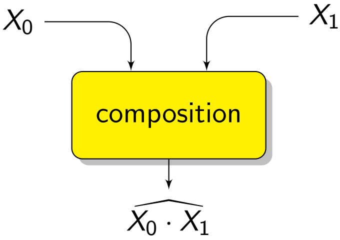
Constants

Constants are **learned** vectors of size n - found by gradient descent

Group implementation

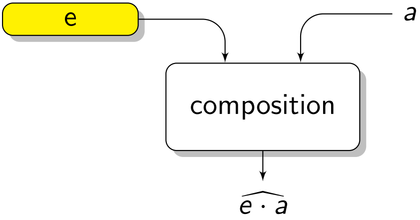
Composition

- Learned as a lookup table
- Some (up to 10%) values missing to test the ability to generalize
- Minimizing the squared difference



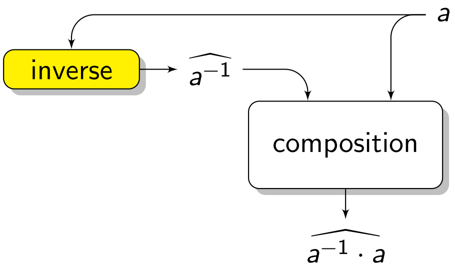
Unit

- Learned from the axiom $e \cdot a = a$
- Used the learned NN for composition and mean squared difference



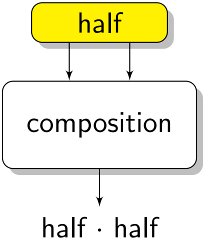
Inverse

- Learned from the axiom $a^{-1} \cdot a = e$
- Used the learned NN for composition and the learned unit element.



Extension

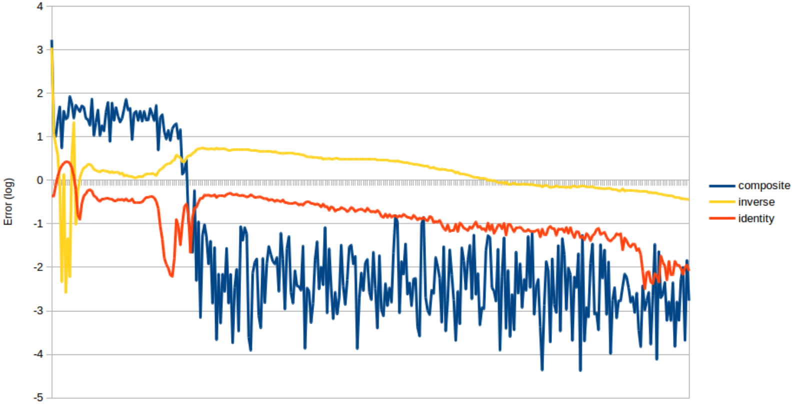
- "What does half look like"
- Using the learned composition we find a constant h such that $h + h = 1$ (in $\mathbb{Z}_n$) or $h \circ h = (1, 0)$ (in S_n)
- This h is **not** in the original embedding
- We look at the relationships between h and original elements



Section 3

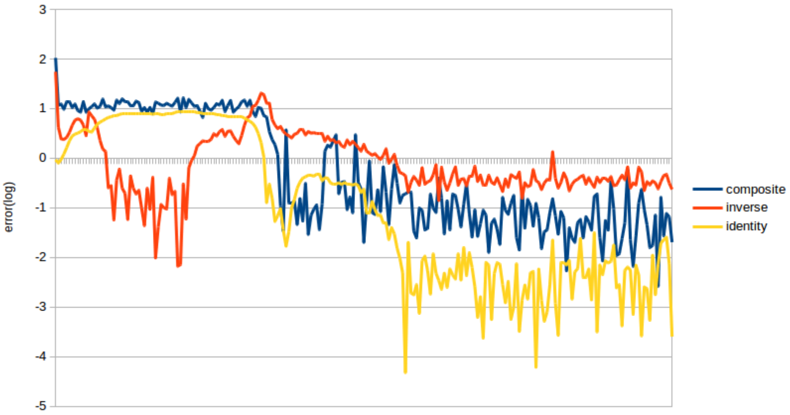
Results

$\mathbb{Z}_{10}$ with 10% testing data



sample (learned) composition: $\widehat{8+8} = 6.048121$; $\hat{e} = 0.00823911$

$\mathbb{Z}_{20}$ with 10% testing data



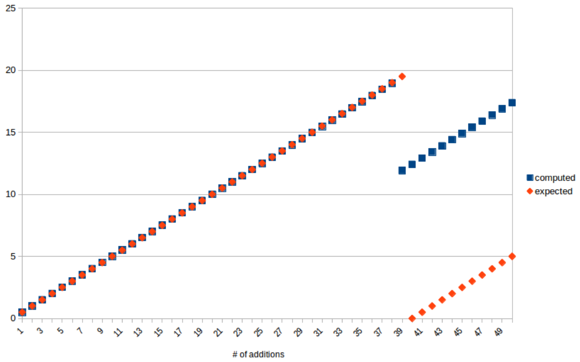
$\mathbb{Z}_{20}$ half training

Values for half in different runs:

0.4999506	-6.5685954	10.500707	0.49987993	0.5000777
0.49967808	0.49978873	0.49993014	0.50047106	10.499506

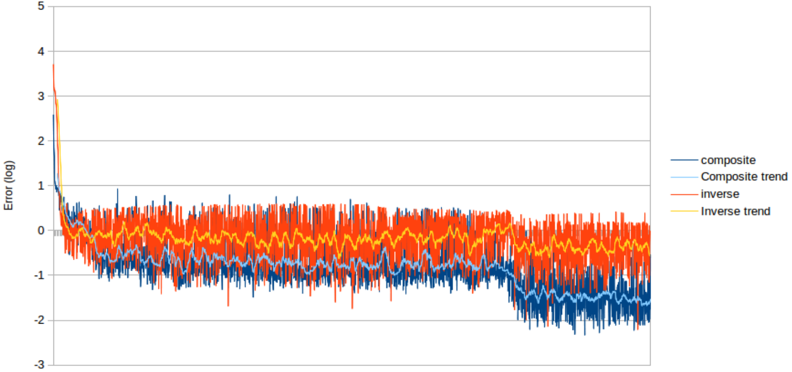
Group generated by learned half

We generate the group $\mathbb{Z}_{40}$ by using the learned composite on learned half repeatedly.



Permutation group S_4

Basic embedding



Identity: 0.0015894736 1.0011026 2.0010371 2.9997234

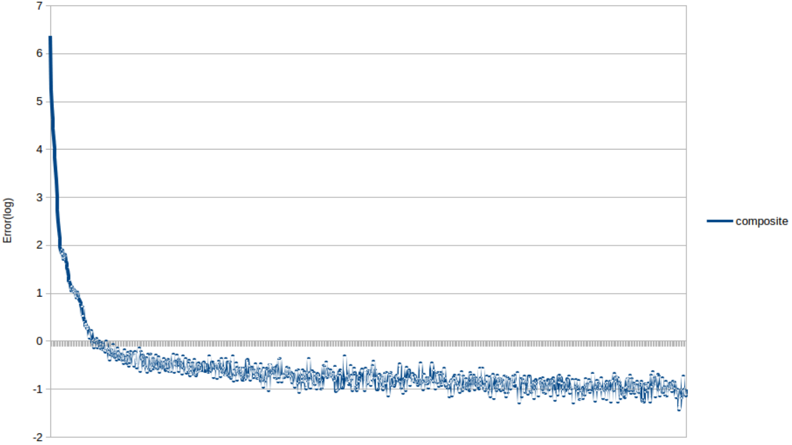
S_4 half

Basic embedding

h:	3.0214617	1.5137568	0.34816563	1.1509237
$h \circ h$:	1.007834	-0.00332985	2.0015383	2.9760256
h^4:	3.133353	-0.3489724	1.1988858	2.931558

Permutation group S_4

one-of-n representation



S_4 identity

one-of-n representation

0.9291287	0.39432997	-0.09073094	-0.00462165
-0.49930313	2.5813785	-1.0001388	-0.51179993
0.38528794	0.32126167	1.4879085	-0.3122037
0.16110185	-0.10436057	-0.3780875	1.356762

S_4 half

one-of-n representation

learned half element:

-0.44275028	0.45813385	0.84849375	-0.4929848
0.29338264	0.25557452	0.701611	-0.33617198
0.5497755	0.75910103	-0.16280994	-0.17575327
0.20515643	0.25966993	0.10358979	1.0272595

S_4 half

one-of-n representation

$h \circ h$:

0.0016880417	0.99703968	-0.0002135747	-0.0009868203
0.99901026	-0.0018832732	-0.0010174632	0.0011361403
-0.0027710588	-0.0013556076	1.0073379	-0.001112761
-0.0005431428	0.0049326816	-0.0010595275	1.00323

Very nice!

S_4 half

one-of-n representation

$h \circ (h \circ (h \circ h))$:

0.72058374	0.17218184	0.04241377	0.04261543
0.4558762	-0.00405501	0.55539876	0.00293861
-0.02832983	0.10163078	0.4973646	0.4502491
-0.02180864	0.6911213	-0.02542126	0.4643306

What the $*%& is that?!

This is not identity!!

Section 4

Comments

Comments

- More time/ power
- Relations
- Self-found embeddings
- Infinite structures
- $\exists$