EQUATION WRANGLING: HOW I LEARNED TO STOP WORRY ABOUT LARGE EQUATIONAL SETS AND LOVE AUTOMATED DEDUCTION

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A *quasigroup* is a set with a binary operation, $\cdot$, such that in $x \cdot y = z$, knowledge of any two of x, y and z specifies the third uniquely [5]. This definition is combinatorialy inflected, easy to grasp and of great value, but alas, not equational, so of little use or interest to an automated theorem user. Happily, there is an equivalent definition that is entirely equational—a *quasigroup* is a set with three binary operations, $\backslash$, $/$ and $\cdot$, satisfying the following four identities :

$$x \cdot (x\backslash y) = y,\ (y/x) \cdot x = y,\ x\backslash(x \cdot y) = y \text{ and } (y \cdot x)/x = y.$$

A *loop*, then, is a quasigroup with a (unique) two-sided neutral element, denoted here by e.

As a notational simplification, we write xy instead of $x \cdot y$, and reserve $\cdot$ to have lower priority than juxtaposition among factors to be multiplied; for instance, $(x \cdot xy)z = (y \cdot zx)x$ stands for $(x \cdot (x \cdot y)) \cdot z = (y \cdot (z \cdot x)) \cdot x$.

In the variety of loops, each of the following four identities implies the other three: $z(xy \cdot z) = zx \cdot yz$, $z(x \cdot zy) = (zx \cdot z)y$, $(z \cdot xy)z = zx \cdot yz$, and $(xz \cdot y)z = x(z \cdot yz)$. A loop that satisfies any one (hence, all four) of these identities is called a *Moufang loop*. A *left Bol loop* is a loop satisfying the identity $x(y \cdot xz) = (x \cdot yx)z$; *right Bol loops* satisfy the mirror identity. The left Bol identity generates a different variety of loops than does the right Bol identity. The intersection of the two Bol varieties is the variety of Moufang loops. These three varieties are amongst the most prominent and intensively investigated varieties of loops.

The six equations axiomatizing them share a number of common features:

(1) they contain only one operation—the loop product,
(2) there are exactly three distinct variables appearing on each side of the equal sign, one appearing twice on each side of the equal

sign, the other two appearing once each on each side of the equal sign, and
(3) the order in which the variables appear in is the same on each side of the equal sign.

An identity that satisfies these three conditions is thus called an identity of *Bol-Moufang type*. It is straightforward to count that there are 60 identities of Bol-Moufang type. There are precisely 14 varietes of loops axiomatized by a single identity of Bol-Moufang type [4], and there are 13 varieties of (not necessarily associative) rings axiomatized by a single identity of Bol Moufang type [6]. We refer to these varieties as *varieties of loops (resp., rings) of Bol-Moufang type*.

By dropping the third condition in the definition of Bol-Moufang type we obtain the following definition: an identity is said to be of *generalized Bol-Moufang type* if it satisfies the following two conditions:

(1) it contains only one operation—the loop product, and
(2) there are exactly three distinct variables appearing on each side of the equal sign, one appearing twice on each side of the equal sign, the other two appearing once each on each side of the equal sign.

A simple count shows that there are 1215 identities of generalized Bol-Moufang type. It is thus a large and nontrivial problem to determine how many varieties of loop are axiomatized by a single identity of generalized Bol-Moufang type. It is also a problem perfectly suited to automated deduction. Thus, one can show that there are precisely 48 varieties of loops axiomatized by a single identity of generalized Bol-Moufang type, including the 14 varieties of Bol-Moufang type [2].

Beyond the quirky satisfactions akin to those of the stamp collection, a classification like this can shed light on heretofore overlooked varieties of loops with unique and compelling structure. In this talk, we describe the structure of three such varieties, Cheban loops [1], Frute loops [3], and a completely new variety of loops called Mate loops (this name will need to be changed) [2].

The emphasis throughout is on automated deduction (in this case, specifically Prover9), especially its utility in dealing with large equational problems.

References

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