Towards a Theory of Affine and Polynomial Biquandles: Enumeration, Ring-Dependence, and Beyond

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Abstract

We investigate finite biquandle structures arising from affine and polynomial constructions over finite rings. Using automated finite model generation and subsequent symbolic analysis, assisted by interactive LLM-guided conjecture formation, we obtain explicit enumeration results for affine realisations of biquandles over fixed finite rings, identify a ring-relative notion of affineness, and demonstrate the existence of polynomial but non-affine structures over $\mathbb{Z}_n$. The results illustrate how automated reasoning, combined with interactive analysis, can uncover new structural phenomena in algebra.

Introduction

Biquandles are algebraic structures associated with knot theory, providing invariants under Reidemeister moves via two binary operations $\overline{\triangleright}$ and $\underline{\triangleright}$ [2, 1]. They generalize quandles [1] and are widely used in the study of knots and generalizations of knots [5, 6].

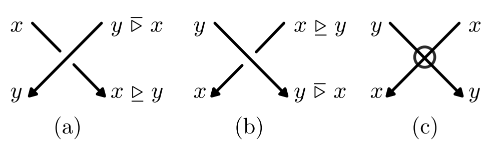

Figure 1: Three local crossing configurations — (a) and (b) classical crossings, and (c) a virtual crossing — with incoming and outgoing arcs labelled to illustrate the corresponding biquandle operations

A prominent and widely used class is that of *affine biquandles*, where both operations are defined over a ring R by linear expressions $ax + by + c$, where $a, b, c \in R$. Despite their simple form, these constructions exhibit rich combinatorial behavior, which we study by combining automated reasoning methods with LLM[1] assisted human analysis. This work is partially connected to earlier studies on applications of automated reasoning to knots via quandle-based methods [3, 4]. A version of this extended abstract with Appendices A–C is available as supplementary material on Zenodo [2].

Enumeration via automated reasoning

Affine biquandles were generated by encoding the defining axioms (see Appendix A of the supplementary material) as first-order formulas, using f and g for the operations $\overline{\triangleright}$ and $\underline{\triangleright}$, respectively, and applying finite model generation with `Mace4`.

[1]In this study we used ChatGPT based on the GPT 5.3 model
[2]https://zenodo.org/records/22024566

In this enumeration, $A(n)$ counts affine realisations over the standard ring $\mathbb{Z}_n$, that is, operation pairs f, g defined by affine terms on the fixed ring structure $\langle \mathbb{Z}_n, +, \times, 0, 1\rangle$. Distinct affine realisations over this ring may become isomorphic after the ring operations are forgotten, so this is not a count of pure biquandle isomorphism types.

Initial values of $A(n)$ were first obtained by explicit enumeration of models over $\mathbb{Z}_n$ with `Mace4`. These data were then analysed symbolically by translating the affine biquandle axioms into constraints on the affine coefficients. This constraint analysis, supported by interactive LLM-assisted pattern recognition and followed by direct symbolic verification, led to the following closed formula in the prime case.

Proposition 1. *Let p be a prime. The number of affine biquandle realisations over $\mathbb{Z}_p$ is*

$$A(p) = p(p-1)(2p-3).$$

For composite n, the same translation of the axioms into coefficient constraints gives a semi-explicit decomposition derived from the Chinese Remainder structure of $\mathbb{Z}_n$. This allows extraction of explicit constraints on the affine coefficients and efficient computation of $A(n)$. The values obtained in this way were cross-validated against the direct `Mace4` enumerations. The values $A(n)$ for n=3..9 are tabulated in Appendix B of the supplementary material.

Ring-relative affineness

The enumeration results reveal that affineness depends essentially on the chosen ring.

Proposition 2. *There exist finite biquandles which are affine over one ring of order n but not affine over another ring of the same cardinality.*

This phenomenon is illustrated by the complete classification of biquandle isomorphism types for rings of size 4:

$$\mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{F}_2[\varepsilon]/(\varepsilon^2), \quad \mathbb{F}_4.$$

In this classification, we consider abstract biquandles up to isomorphism, disregarding the underlying ring in which an affine realization is defined. Across these rings, there are 28 distinct isomorphism types of affine biquandles, distributed as follows: a) The 2 types are realized on all four rings; b) the 17 types are realized on exactly two rings, namely the $\mathbb{Z}_4$ and $\mathbb{F}_2[\varepsilon]/(\varepsilon^2)$; c) the 9 types are realized only on $\mathbb{F}_4$.

This demonstrates that affineness is not an intrinsic property of a biquandle, but depends on the algebraic structure of the underlying ring.

Polynomial vs affine constructions

Affine constructions form a special subclass of polynomially definable biquandles. Over prime rings $\mathbb{Z}_p$, every binary operation $\mathbb{Z}_p^2 \to \mathbb{Z}_p$ is represented by a polynomial in two variables; hence polynomial definability is much broader than affineness, but imposes no restriction on finite operation tables. Over composite rings such as $\mathbb{Z}_n$, not every binary operation is polynomial, so polynomial definability becomes a nontrivial intermediate condition between affine definability and arbitrary finite operations.

Proposition 3. *There exist polynomially definable biquandle structures over $\mathbb{Z}_4$ and $\mathbb{Z}_6$ which are not affine.*

Illustrative examples are provided in Appendix C of the supplementary material. These structures were identified using automated reasoning combined with congruence-based analysis.

Conjecture 1. *There exist polynomially definable but not affine biquandle structures over $\mathbb{Z}_n$ for all composite $n > 2$*

Conclusion

The study combines automated finite model generation with `Mace4`, symbolic analysis of the resulting algebraic constraints, and interactive LLM-assisted exploration. In this workflow, the LLM was used to help inspect generated model data, suggest candidate coefficient constraints and counting patterns, and formulate follow-up computational checks. The resulting formulas, examples, and counts were then verified by direct symbolic calculation and independent model-generation runs. This workflow enables systematic exploration of finite algebraic structures and supports conjecture formation grounded in computational evidence. Future work includes extending these results to larger cardinalities, developing a general theory of affine and polynomial biquandles, and investigating how the observed classification phenomena influence topological applications and the effectiveness of biquandle-based invariants for knotted structures.

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