This talk is centered on automated deduction in mathematical research. We will thus need to develop a bit of the mathematical background necessary to understand this research. The talk will be self-contained, and the mathematical technicalities will be relatively modest. Again, the emphasis is on automated deduction in mathematical research.

A *quasigroup* is a set with a binary operation such that in $x \cdot y = z$, knowledge of any two of x, y and z specifies the third uniquely. A *loop* is a quasigroup with a two-sided identity element. For an overview of the theory of loops, see [1], [2], [11]. Loops, per se, are so general that they resist mathematical analysis; one needs more structure. For example, loops that satisfy the associative law have elegant—even tight—structure: they are groups (hence, loops are often referred to as nonassociative groups).

Loops that satisfy the identity $x \cdot (y \cdot (x \cdot z)) = ((x \cdot y) \cdot x) \cdot z$ are called *Moufang loops.* (We note that there are three other equivalent identities one can use to axiomatize this variety.) The theory of Moufang loops is sophisticated and quite deep, and includes a large collection of highly specialized and technical results [3]. For example, these loops satisfy Sylow's and Hall's theorems, as well as Lagrange's Theorem [5], [6]. An arbitrary loop need not satisfy any of these theorems.

While Moufang loops are highly structured and quite similar to groups, they do, in fact, differ from groups in some fundamental ways. For example, unlike groups, Moufang loops can have nonnormal commutants [7] (definition to follow). This result is part of the long and colorful story (the early parts of which may found in [4]) of classifying/describing those Moufang loops with normal commutants. This story, which is not yet complete, relies heavily on automated deduction.

The *commutant* of a loop, L, is the set of those elements that commute with every element in the loop: $\{x \in \mathrm{L}\colon \forall y \in L, xy = yx\}$. The communtant need not be a subloop [9]. In Moufang loops, though, it is [11]. As above, this subloop (in Moufang loops) need not be normal. But often it is. The pertinent and happy fact for this conference is that the question of which Moufang loops have normal commutant is reducible to an equation, in fact, a not particularly complicated equation. Thus, it is an excellent problem for automated theorem provers. We analyze conditions under which the commutant is normal. The proofs, most of which have been found by automated deduction [10], are complicated (some of them are tens of thousands of lines long and require fairly advanced techniques to find), thus emphasizing the importance of automated deduction.

On a complementary note, we outline a project from the theory of nonassociative rings—a classification that relied on the use of automated theorem provers, but for which the heavy lifting involved the construction of finite models [8].

We will note open problems and ongoing research throughout the talk.

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