Minimal Generating Sets in Magmas

Mikoláš Janota[1] António Morgado[2] Petr Vojtěchovský[3]

[1] Czech Technical University in Prague
[2] IST/INESC-ID, University of Lisbon, Portugal,
[3] University of Denver

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Background

In algebra we have

- groups
- semi-groups
- quasi-groups
- loops
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These are all magmas.

Generating Set (Generators)

- A set S is generating if all the other elements can be obtained by a finite number of multiplications
- **Example:** $\mathbb{N}$ is generated by $\{0, 1\}$ under $+$.

Smallest Generating Sets

$*$	1	2	3
1	2	3	1
2	3	1	2
3	1	2	3

- Is $\{2, 3\}$ generating?

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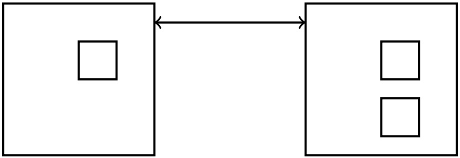
- Is $\{2, 3\}$ generating?... YES
- Is it the smallest possible? NO
- $\{1\}$ is already generating

$$\begin{aligned} 1 &= 1 \\ 1 * 1 &= 2 \\ 1 * 1 * 1 &= 3 \end{aligned}$$

Why Interesting?

Example: Determine if to structures are isomorphic



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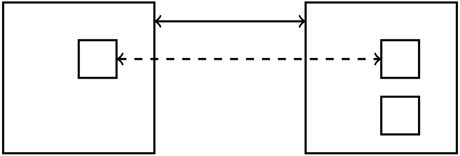

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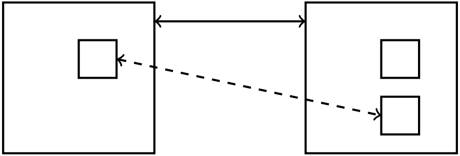

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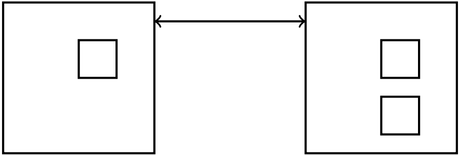

Many other algorithms in computational algebra depend on the generating set and its size.

Calculating Minimal Generating set (finite case)

- The problem is in NP
- SAT solvers scale poorly on the problem
- Can we use SAT without overloading it?

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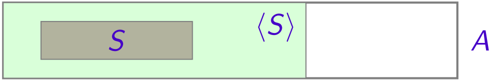

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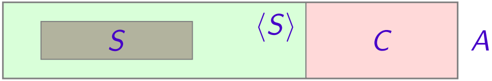

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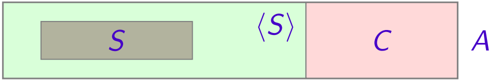

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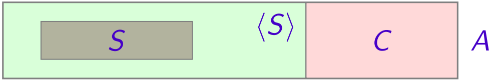

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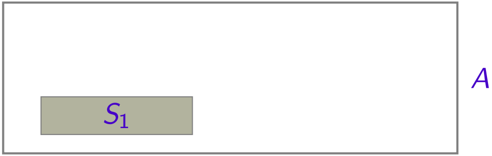
2. If $C \neq \emptyset$, any generating S' must intersect with C.

From Idea to Algorithm



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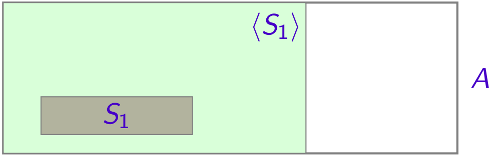

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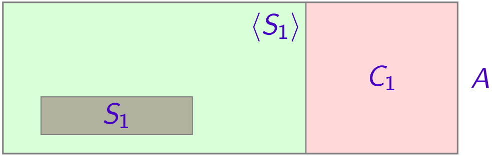

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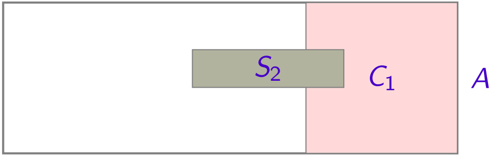

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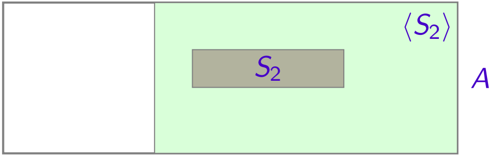

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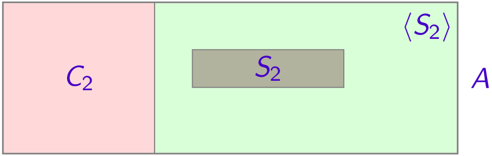

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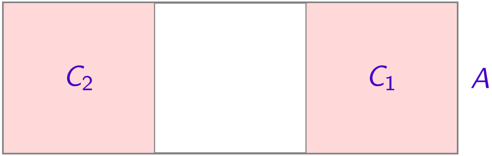

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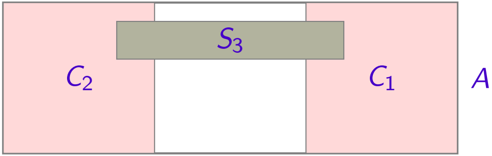

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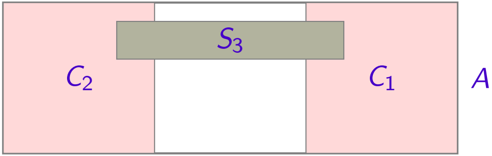

To guarantee the smallest, always calculate the smallest candidate.

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5. We have seen minimal generating sets up to 7

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