ML applications to string theory

FABIAN RUEHLE

AITP 2021

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Based on:

[Anderson, Gray, Gerdes, Krippendorf, Raghuram, FR: 2012.04656]

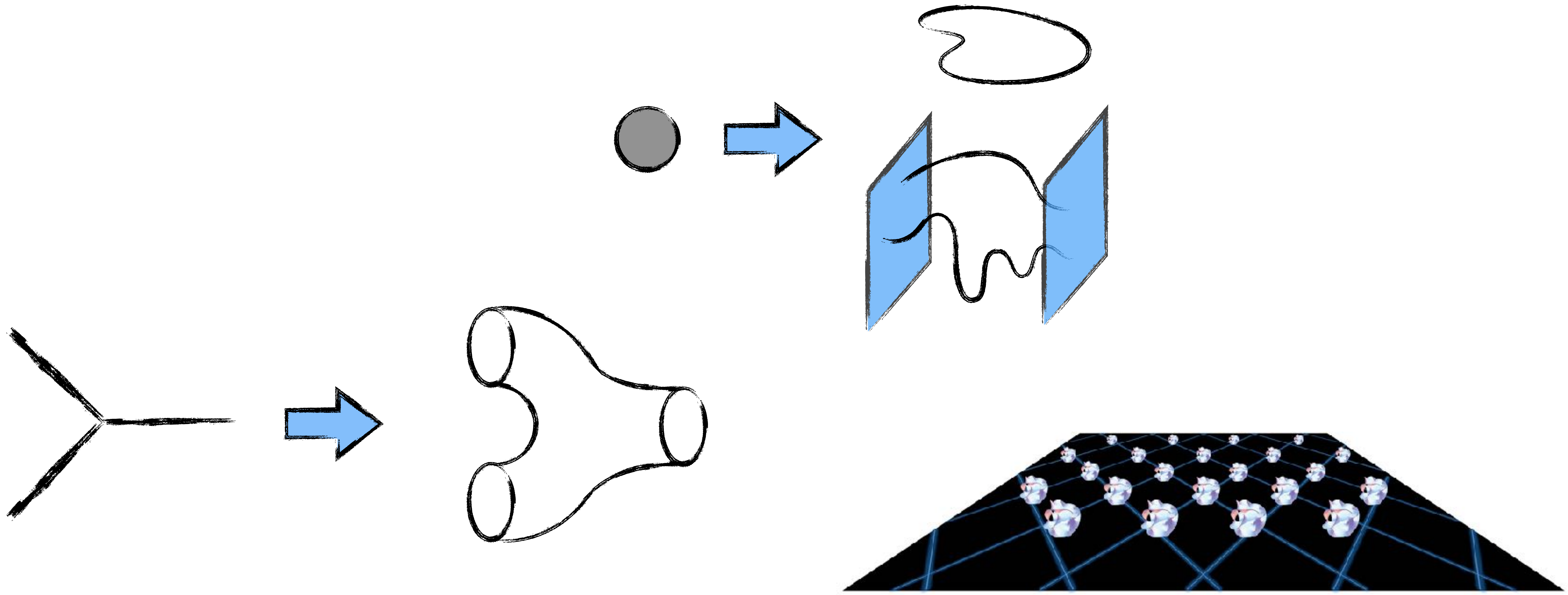
[Gukov, Halverson, FR, Sułkowski: 2010.16263]

[Halverson, Nelson, FR: 1903.11616]

[FR: Physics Reports `20]

Outline

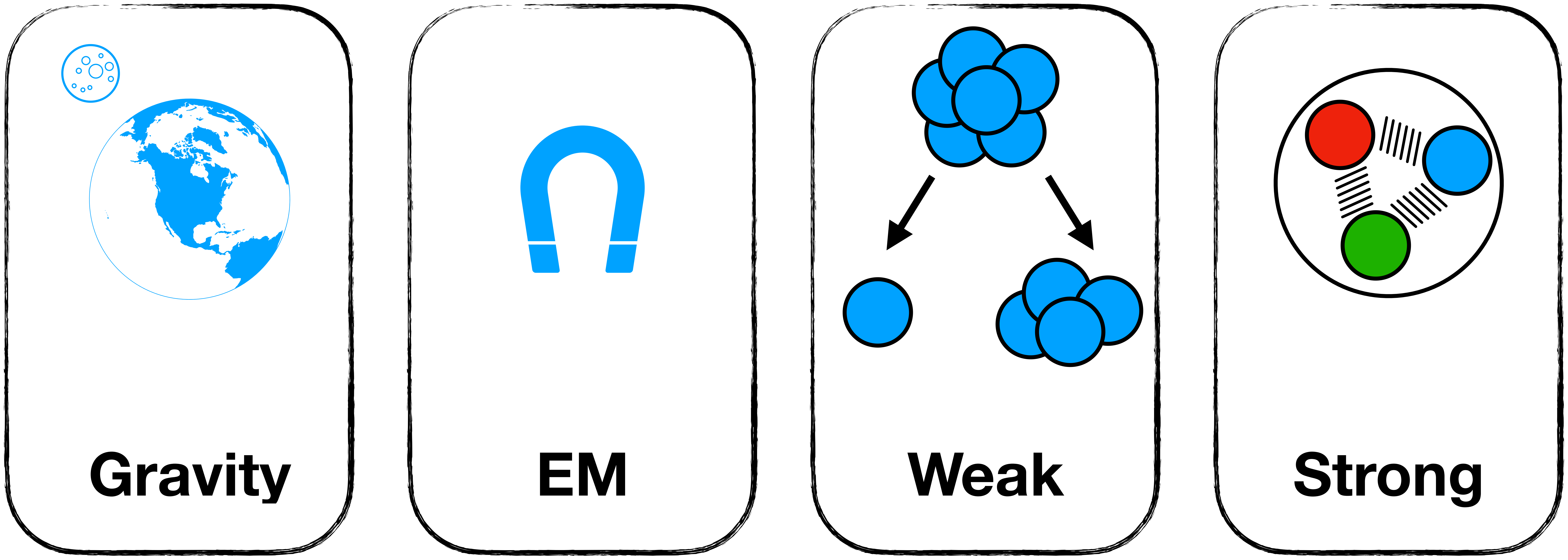
- Introduction to String Theory
- Example applications of ML in String Theory
  - Find solutions to Diophantine equations
  - Find the Unknot
  - Find the Calabi-Yau metric
- Conclusions



Introduction to String Theory

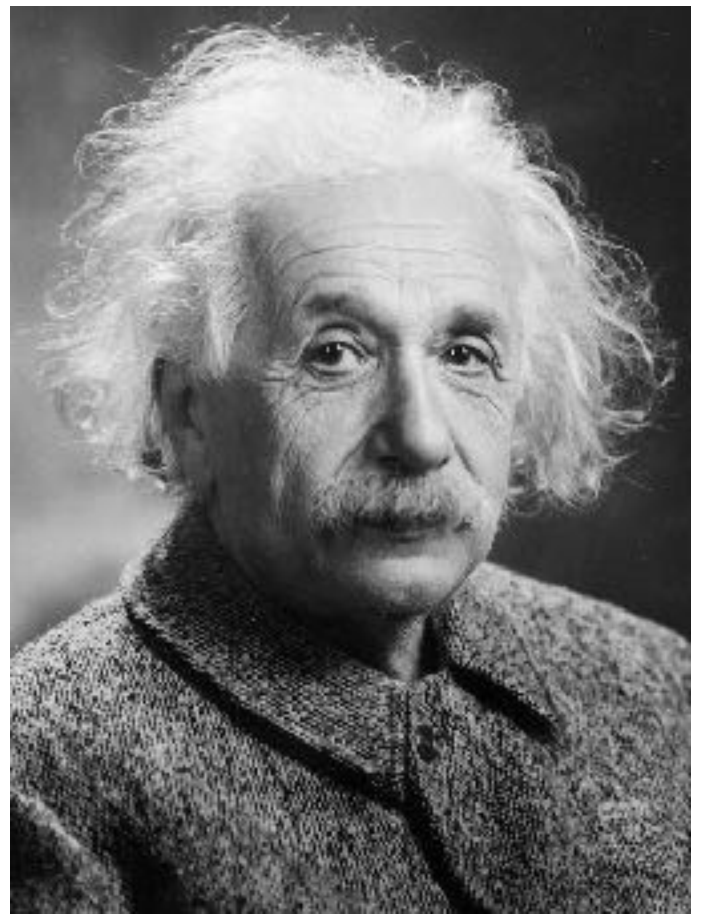
Physics motivation

Many observations in our Universe can be explained with just four fundamental forces

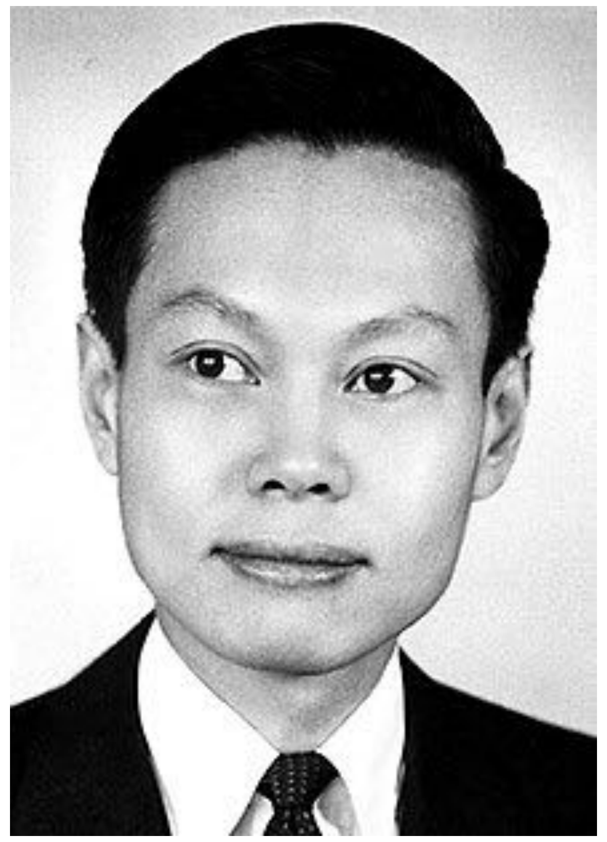


Physics motivation

Two theories to describe these four forces.



General Relativity

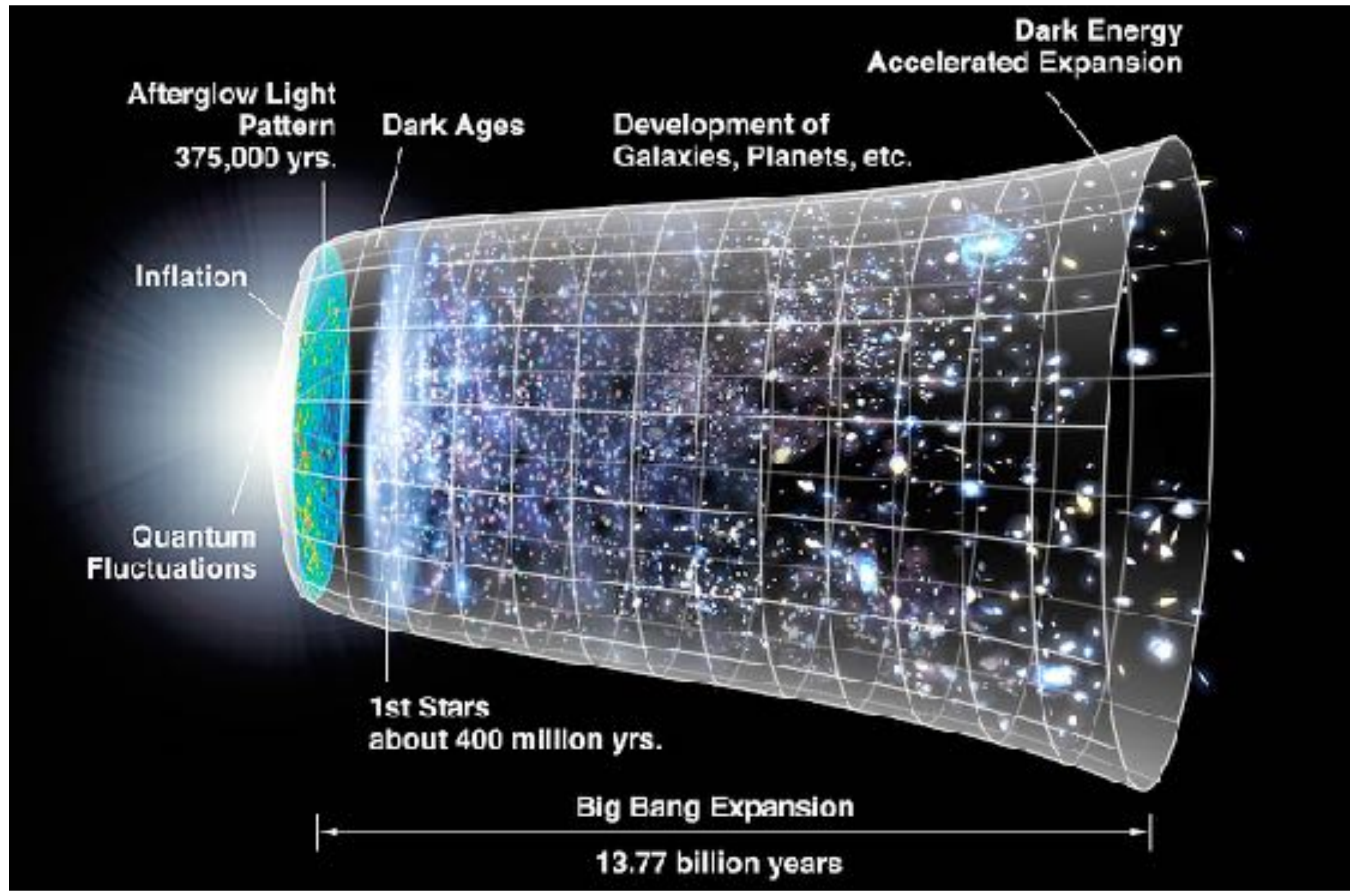


Quantum Field Theory (Yang-Mills Theory)

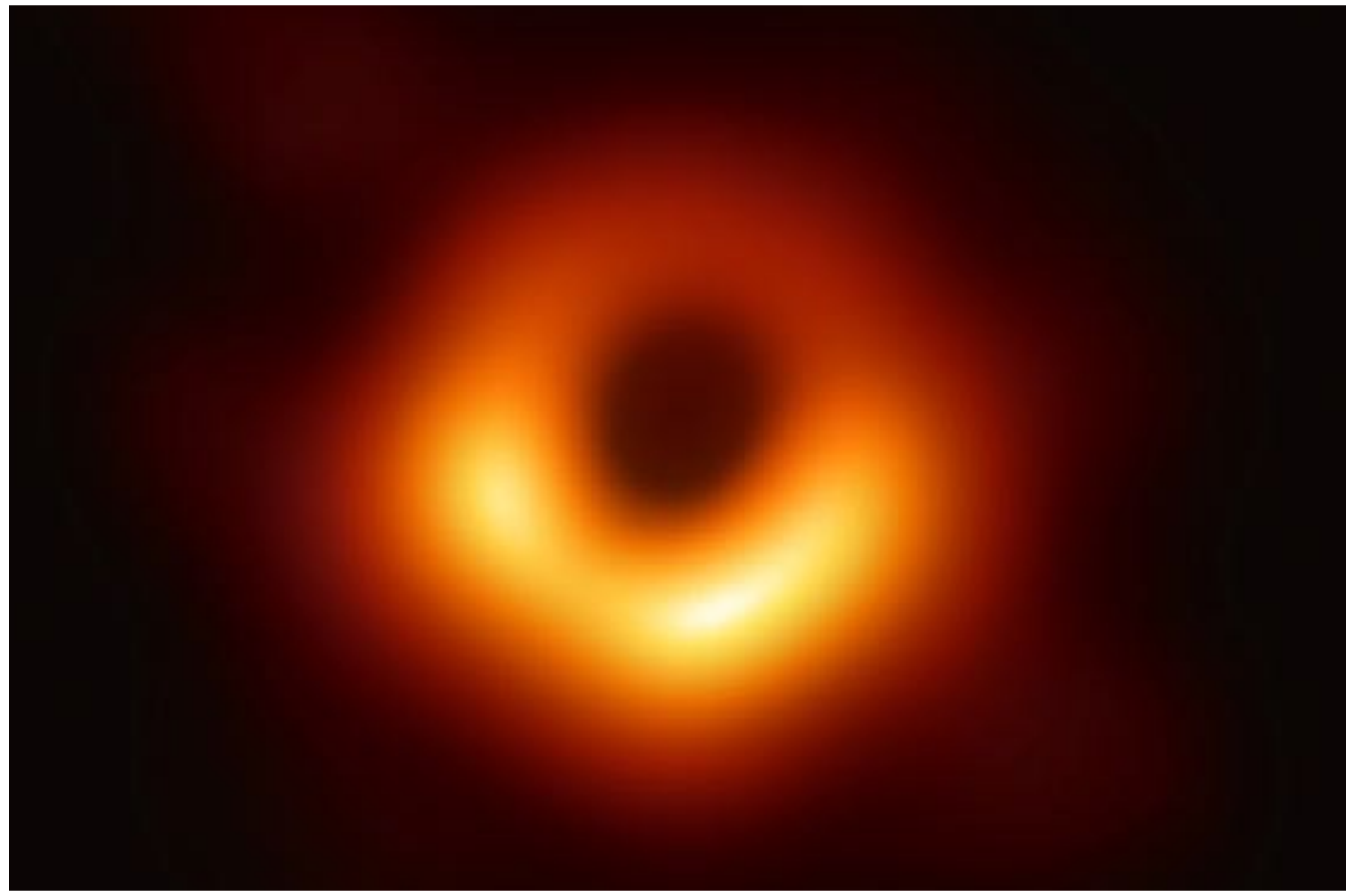
Physics motivation

Physics motivation

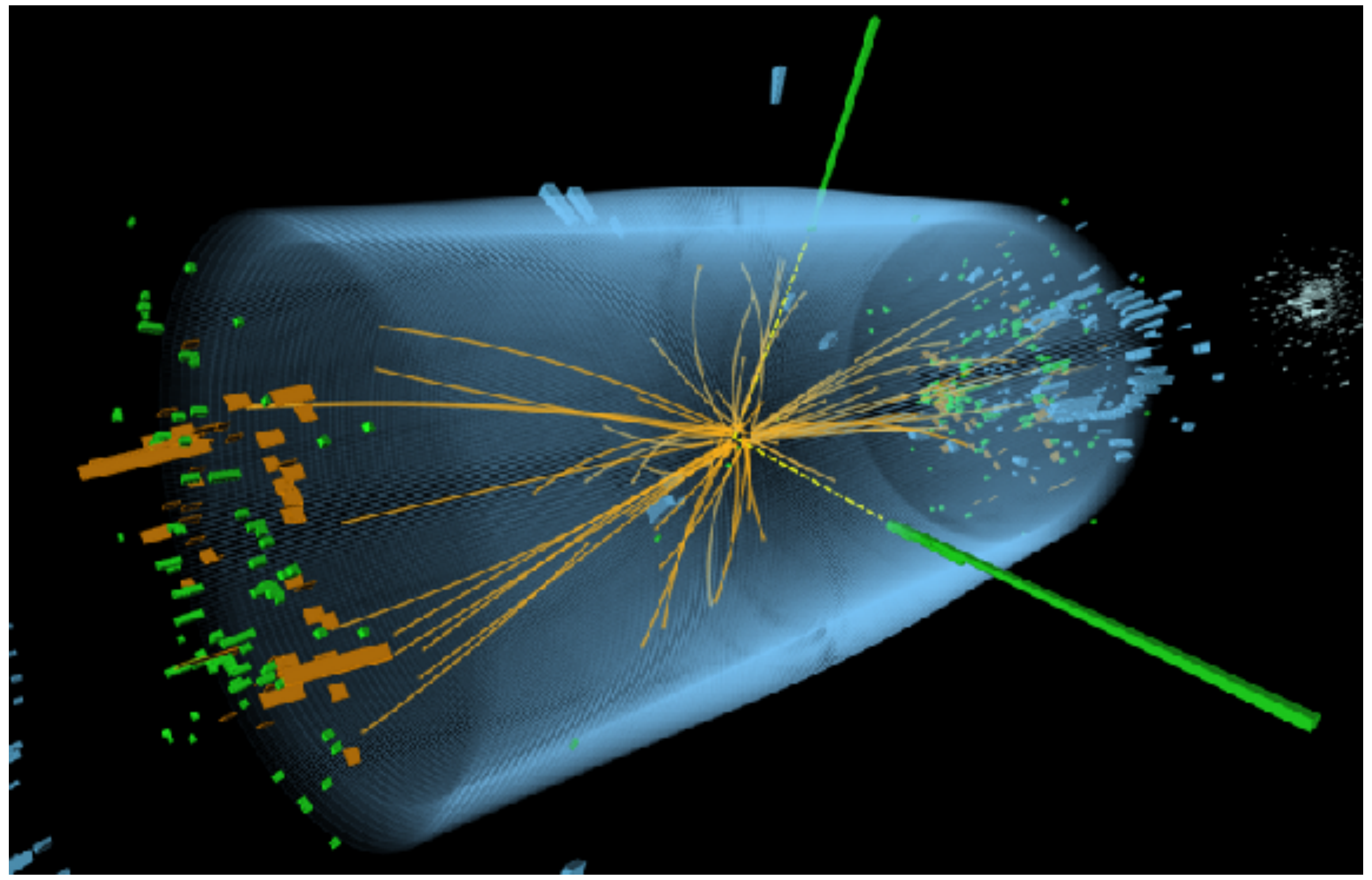
…however, we need a unified description to study physics at high energies



Big Bang, Dark Energy, Inflation



Black Hole Entropy and Information

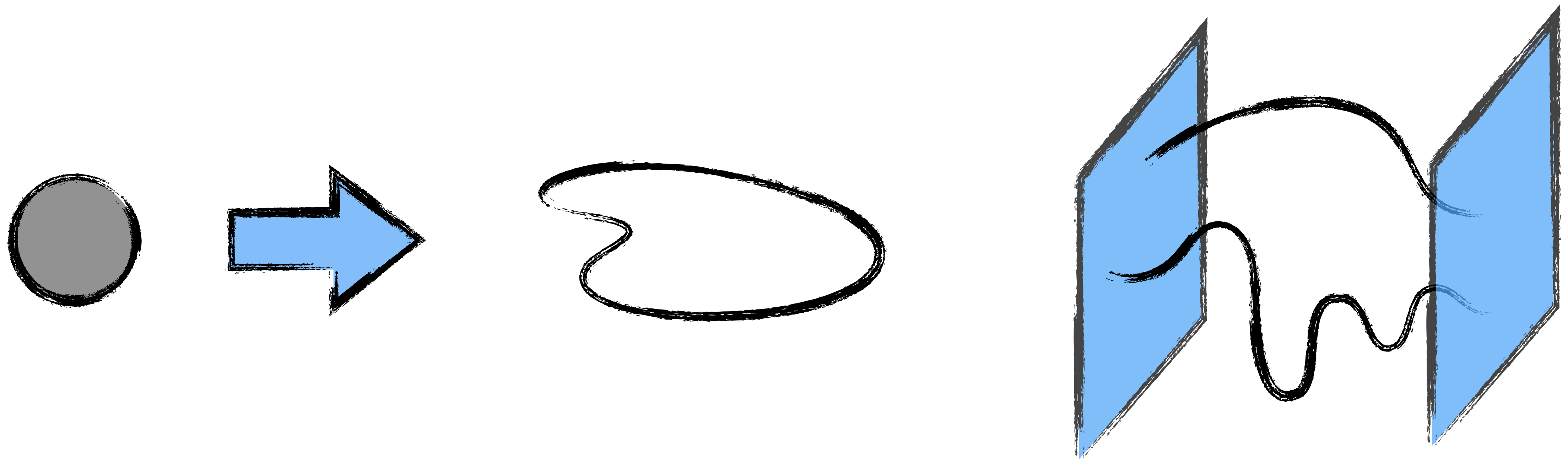


GUT theories and Physics beyond SM

String Theory

- One promising candidate for a unified description of General Relativity and Quantum Field Theory: **String Theory**

- Basic assumption: Fundamental constituents of the particles that mediate the four forces and of all matter are not-point-like, but one-dimensional, extended strings



String Theory - Compactifications

This has far-reaching consequences!

- Stringent constraints:
    - Consistency (mathematical)
    - Match with observed universe (physical)

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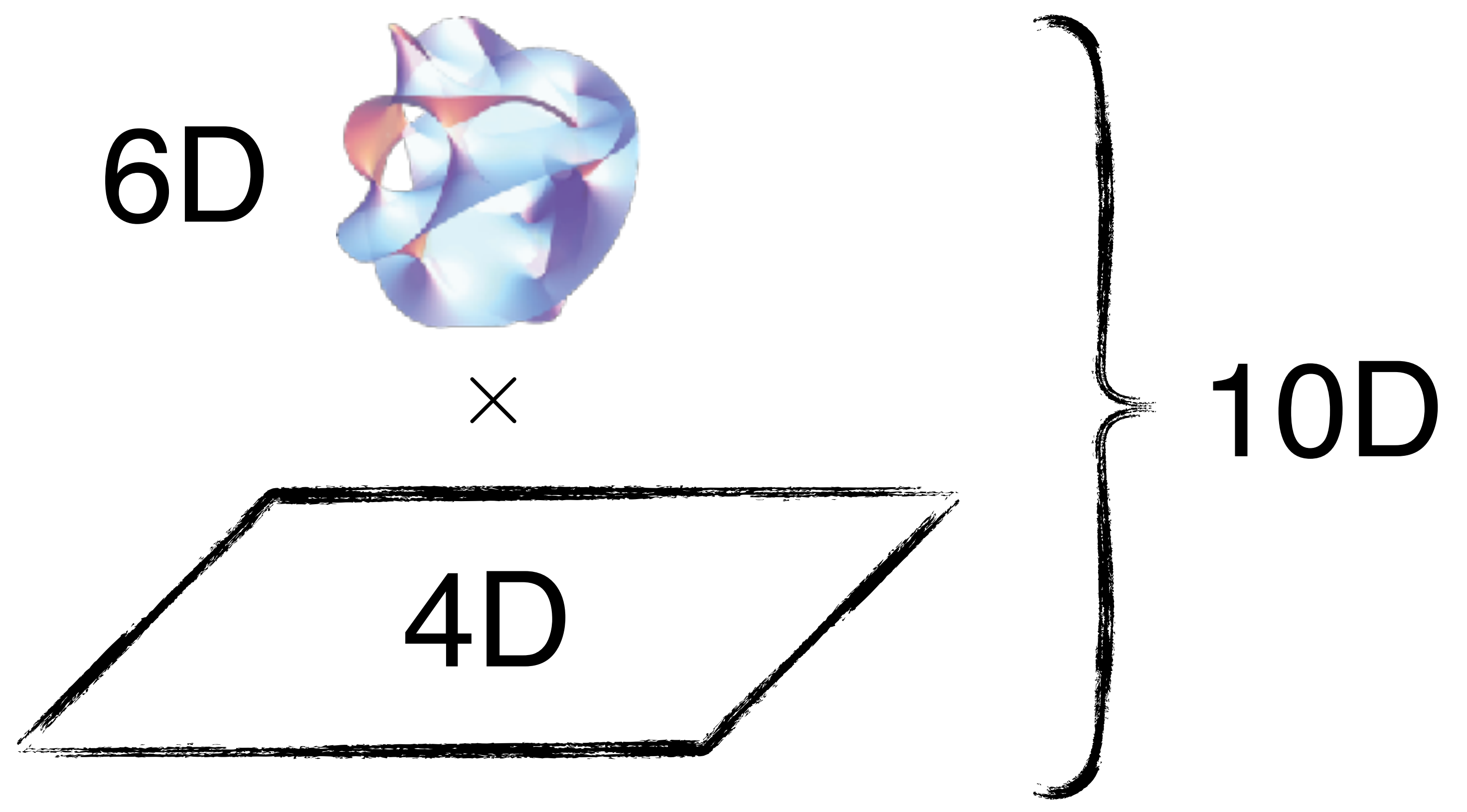
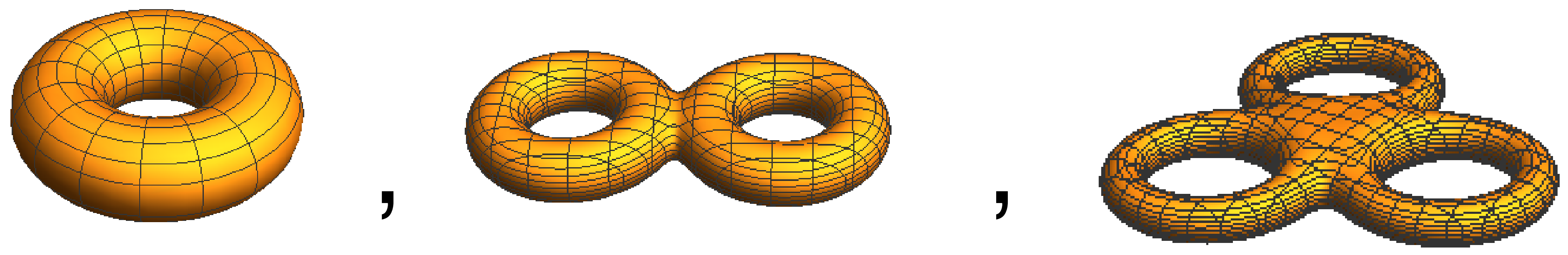
- Stringent constraints:
  - Consistency (mathematical)
  - Match with observed universe (physical)
- Requires ten space-time dimensions
- 10D description essentially unique
- We only observe 3, so 6 have to be small to evade detection ⇒ compactifications



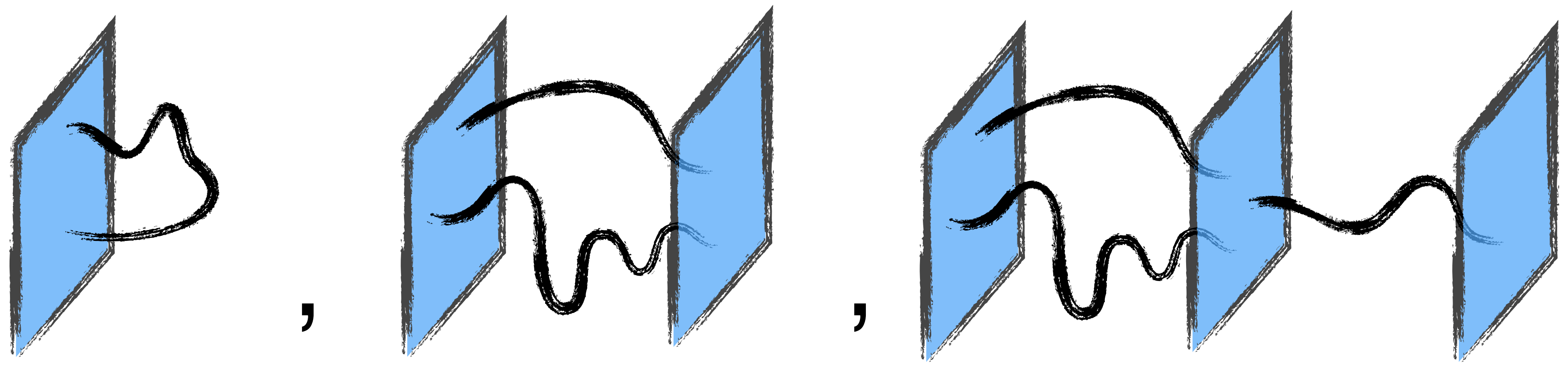
- All of the observable physics is encoded in the 6 compact dimensions (Calabi-Yau manifolds)

String Theory - Compactifications

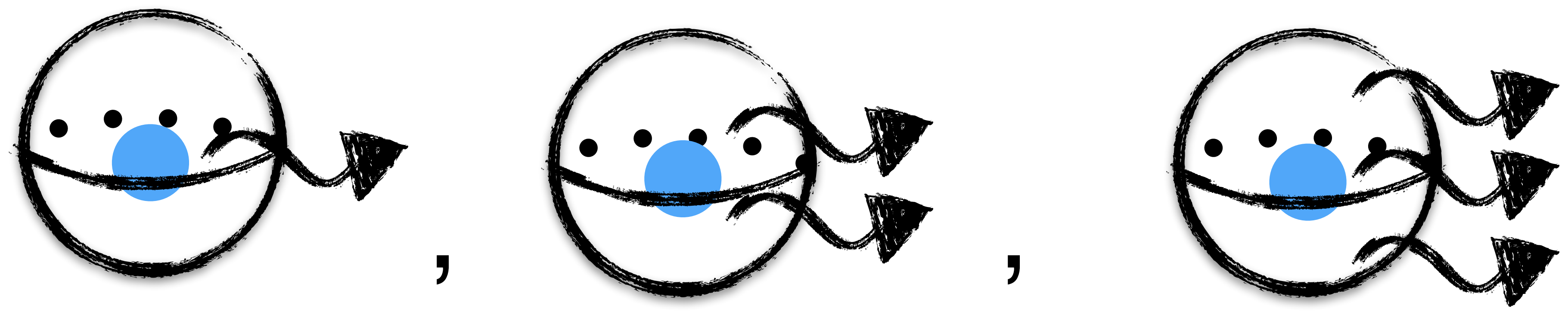
- Discrete data
  - Topology of CY



  - Number of branes



  - Number of fluxes



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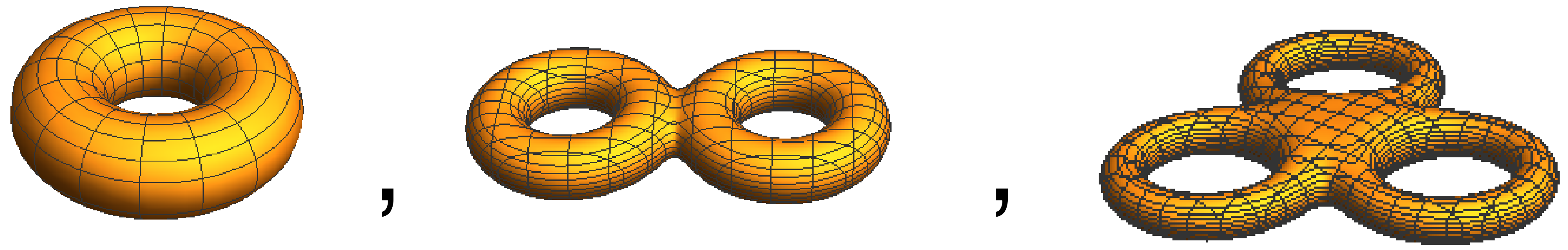

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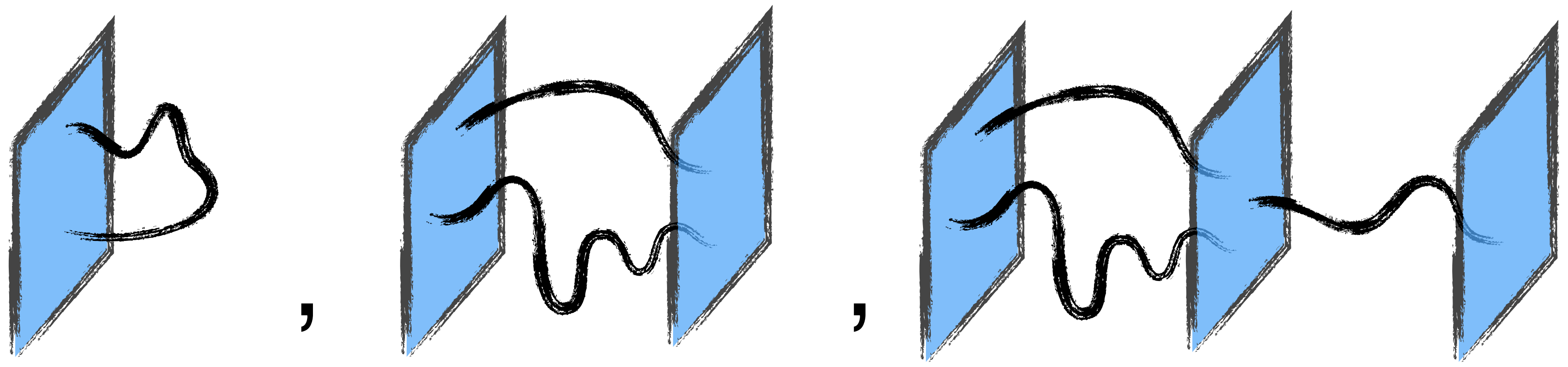

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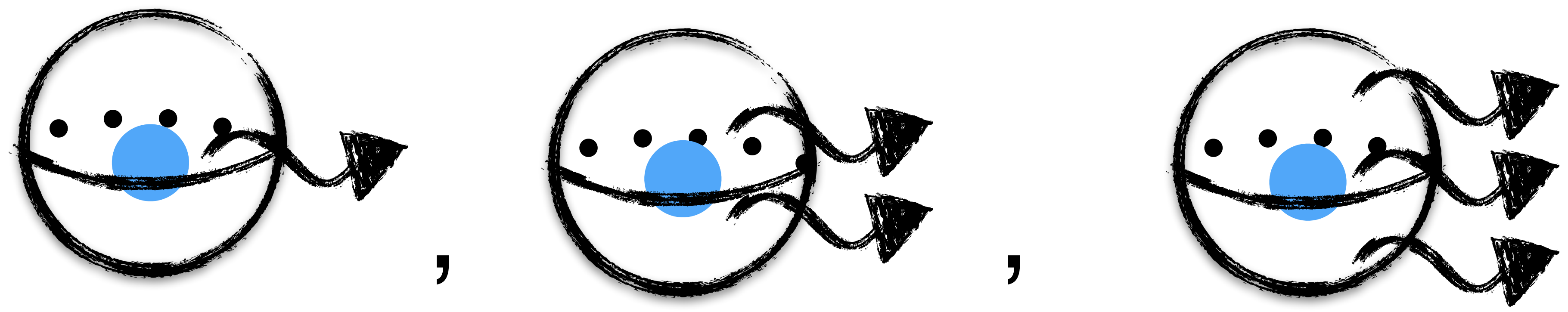
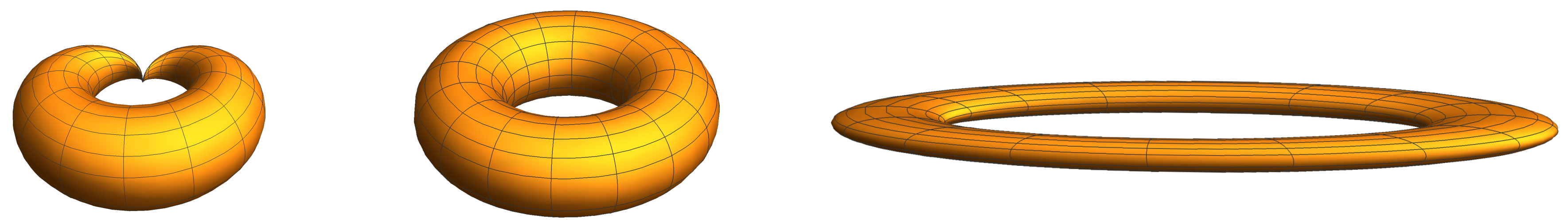
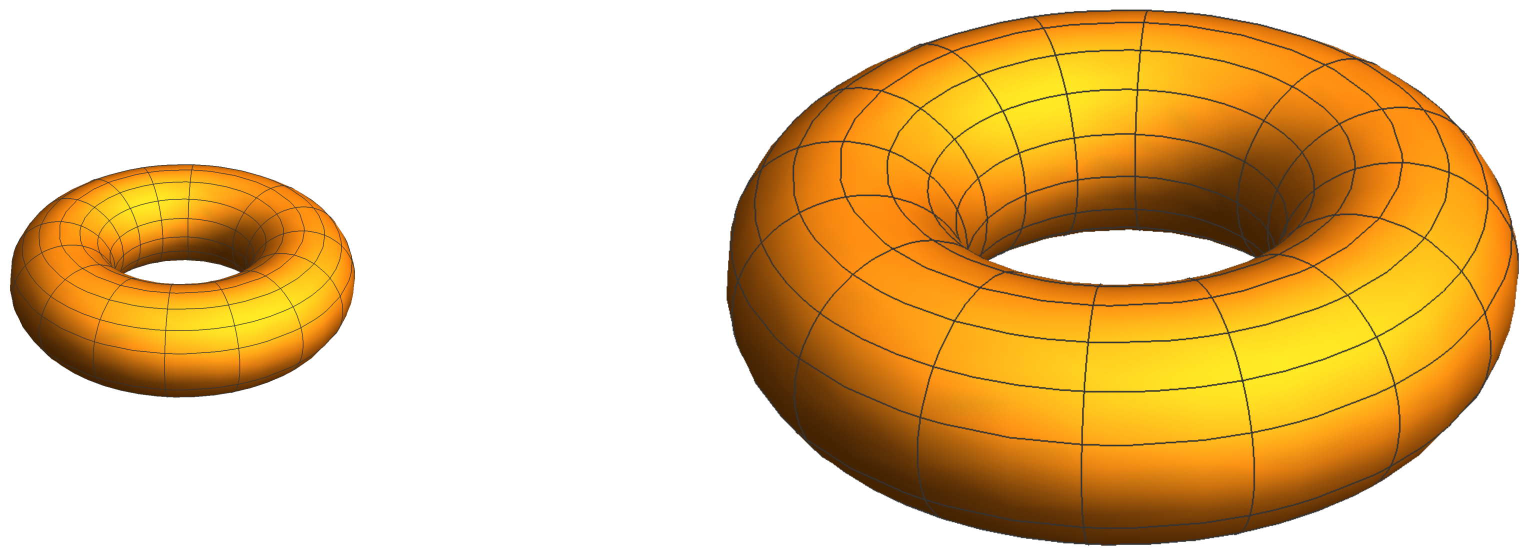


- Continuous data (moduli)
  - Shape of CY



  - Size of CY



ML Applications

Conjecture Generation

Try to learn a map between quantities with no previously known relation, formulate (and hopefully prove) conjecture

- Knot theory
  [Hughes `16; Jejjala,Kar,Parrikar `19; Gukov, Halverson,FR,Sulkowski `20; Craven,Jejjala,Kar `20]
- Toric geometry
  [Krefl,Seong`17;Carifio,Cunningham,Halverson, Krioukov,Long`17]
- Line bundle cohomology, Brill-Noether theory
  [FR `17; Klaewer,Schlechter `18; Brodie,Constantin, Deen,Lukas `18-20; Bies,Cvetič,Donagi,Lin,Liu,FR `20]
- Many more… [especially He et.al.]

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Optimization and Regression

Find solutions to a system of equations

- Searches for string vacua (discrete)
  [FR`17; Wang,Zhang `18; Mutter,Parr,Vaudrevange `18; Halverson,Nelson,FR `19; Brodie,Constantin,Deen, Lukas`19; Larfors,Schneider `20; Deen,He,Lee,Lukas `20; Otsuka,Takemoto `20;Cabo Bizet,Damian,Loaiza-Brito,Mayorga,Montañez-Barrera `20, Constantin, Harvey,Lukas `21]
- CY metrics (continuous)
  [Ashmore,He,Ovrut `19; Anderson,Gray,Gerdes, Krippendorf,Raghuram,FR `20; Douglas, Lakshminarasimhan,Qi `20; Jejjala,Mayorga,Mishra `20]

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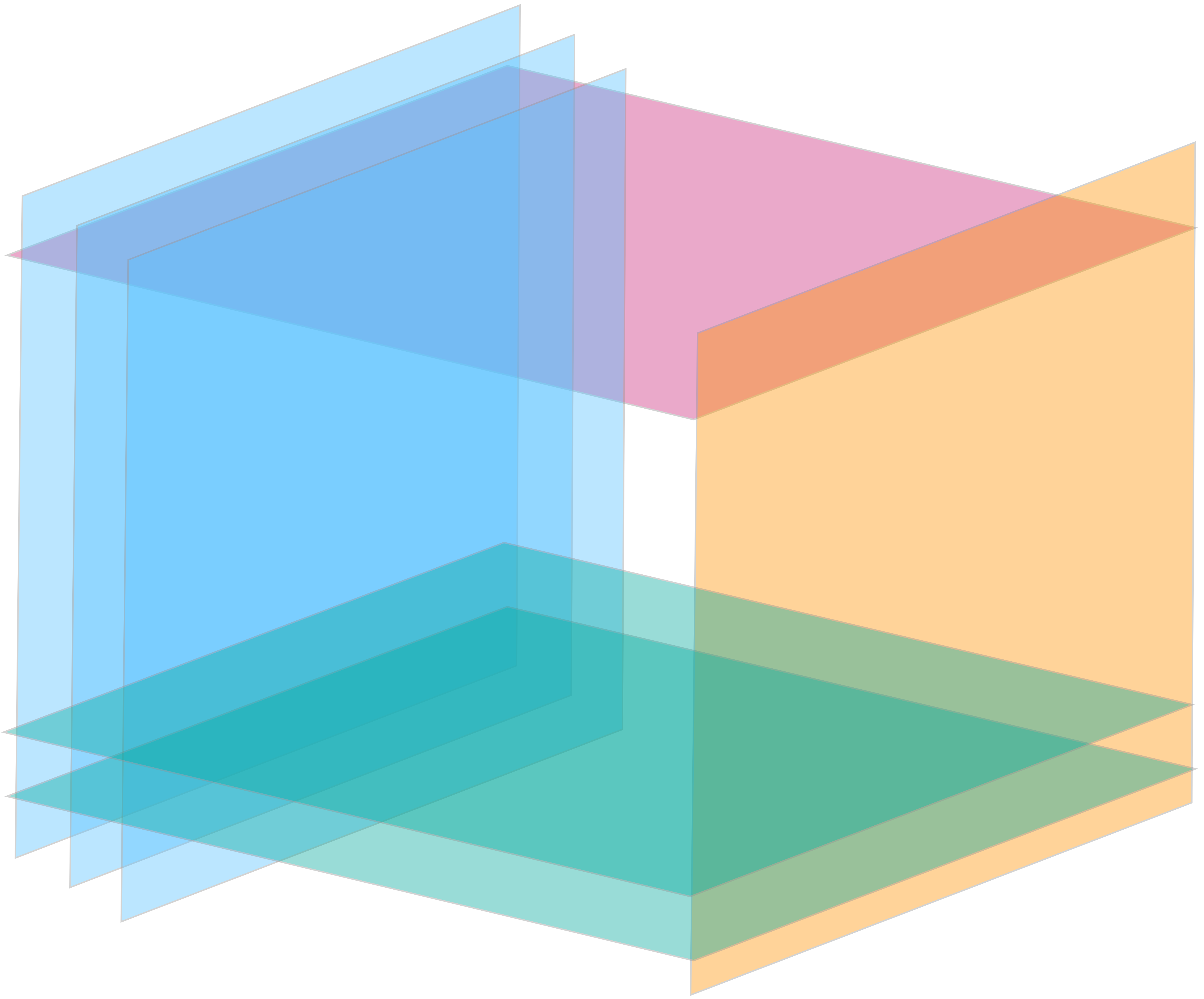
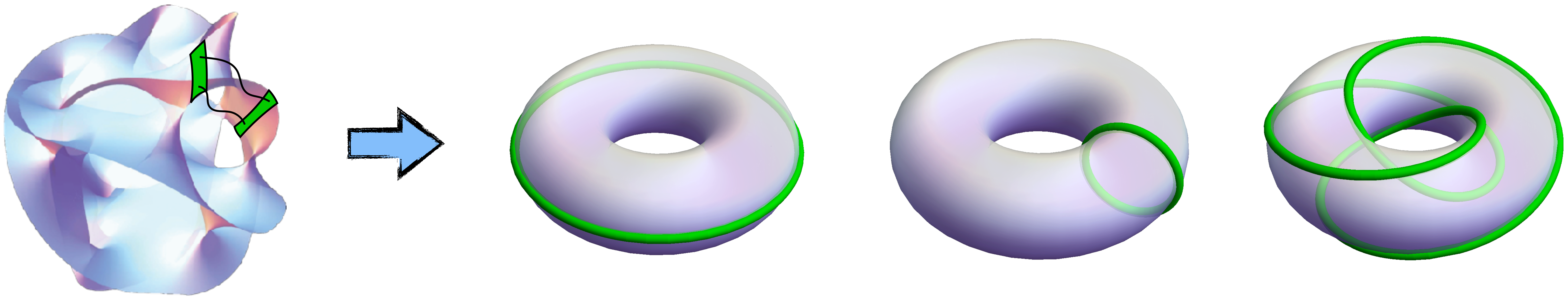


Example I: Solving Diophantine equations

Example I - Solving Diophantine equations

- **Background:**
  - Diophantine equations ubiquitous in ST (topological data, quantization conditions)
  - Asking whether an arbitrary Diophantine equation has a solution (let alone finding one) is undecidable
  - However, Diophantine equations in string theory are not be arbitrary but inherit structure from consistency conditions, …
- **Idea to solve the problem:** [Halverson, Nelson, FR `19]
  - Set up a “game” in RL to solve a particular set of coupled Diophantine equations related to flux vacua of type II orientifolds. This showed that the NN
    - … can rediscover human-derived solution strategy that leads to partial decoupling
    - … can find new, more efficient strategies

First Example - Finding string solutions



- Wrap branes around torus cycles and stack multiple branes on top of each other
- Brane stacks ⇔ Tuple: $(N, n_1, m_1, n_2, m_2, n_3, m_3)$
- There is a finite (but huge) number of inequivalent configurations

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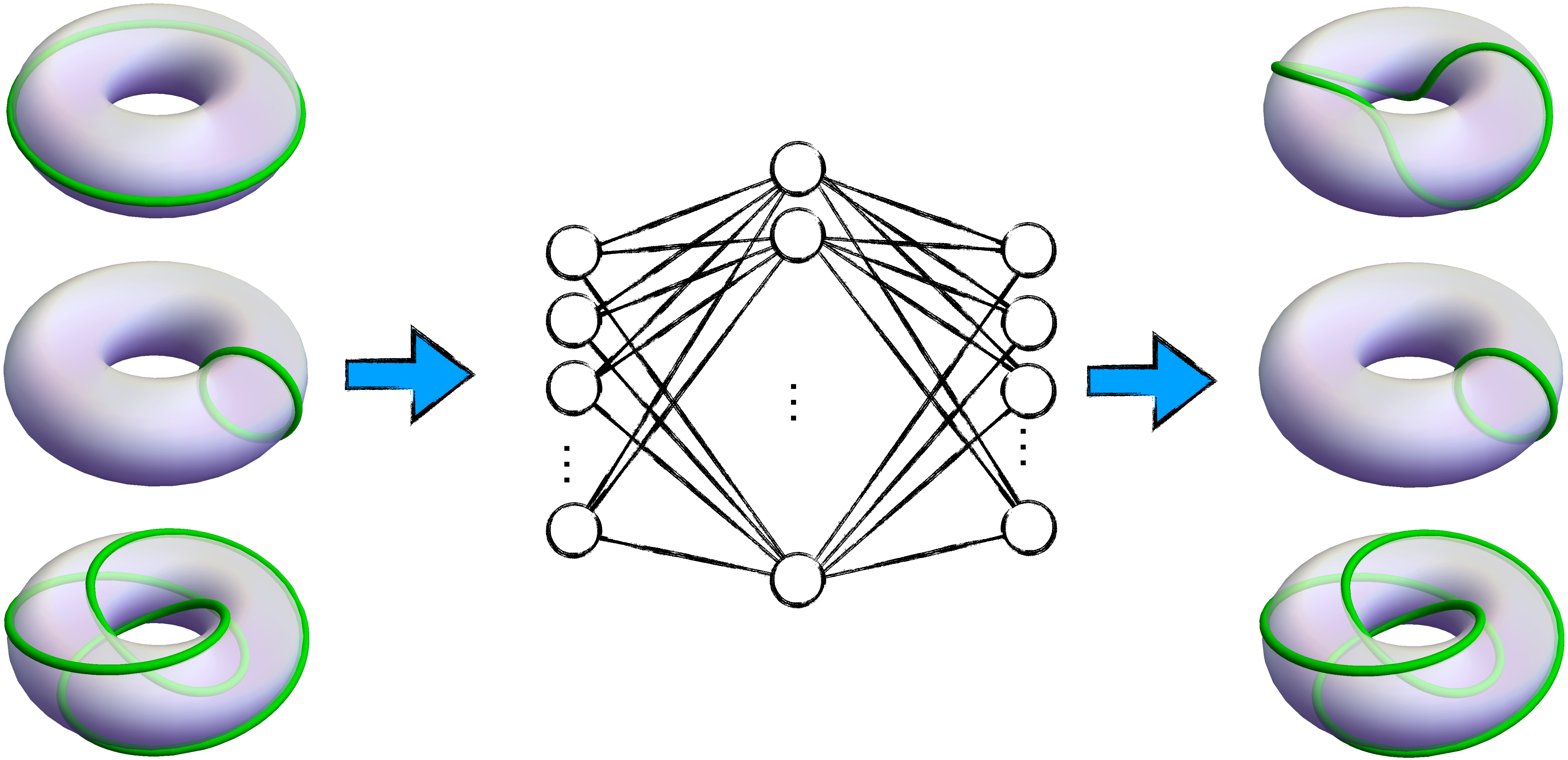

First Example - Finding string solutions

Condition TC:

$$\sum_{a=1}^{\#\text{stacks}} \begin{pmatrix} N^a\, n_1^a\, n_2^a\, n_3^a \\ -N^a\, n_1^a\, m_2^a m_3^a \\ -N^a m_1^a\, n_2^a\, m_3^a \\ -N^a m_1^a m_2^a\, n_3^a \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \\ 8 \end{pmatrix}$$

Condition K:

$$\sum_{a=1}^{\#\text{stacks}} \begin{pmatrix} 2N^a m_1^a m_2^a m_3^a \\ -N^a m_1^a\, n_2^a\, n_3^a \\ -N^a\, n_1^a\, m_2^a\, n_3^a \\ -2N^a\, n_1^a\, n_2^a\, m_3^a \end{pmatrix} \bmod \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Condition S:

$$m_1^a m_2^a m_3^a - j\, m_1^a n_2^a n_3^a - k\, n_1^a m_2^a n_3^a - \ell n_1^a n_2^a m_3^a = 0$$
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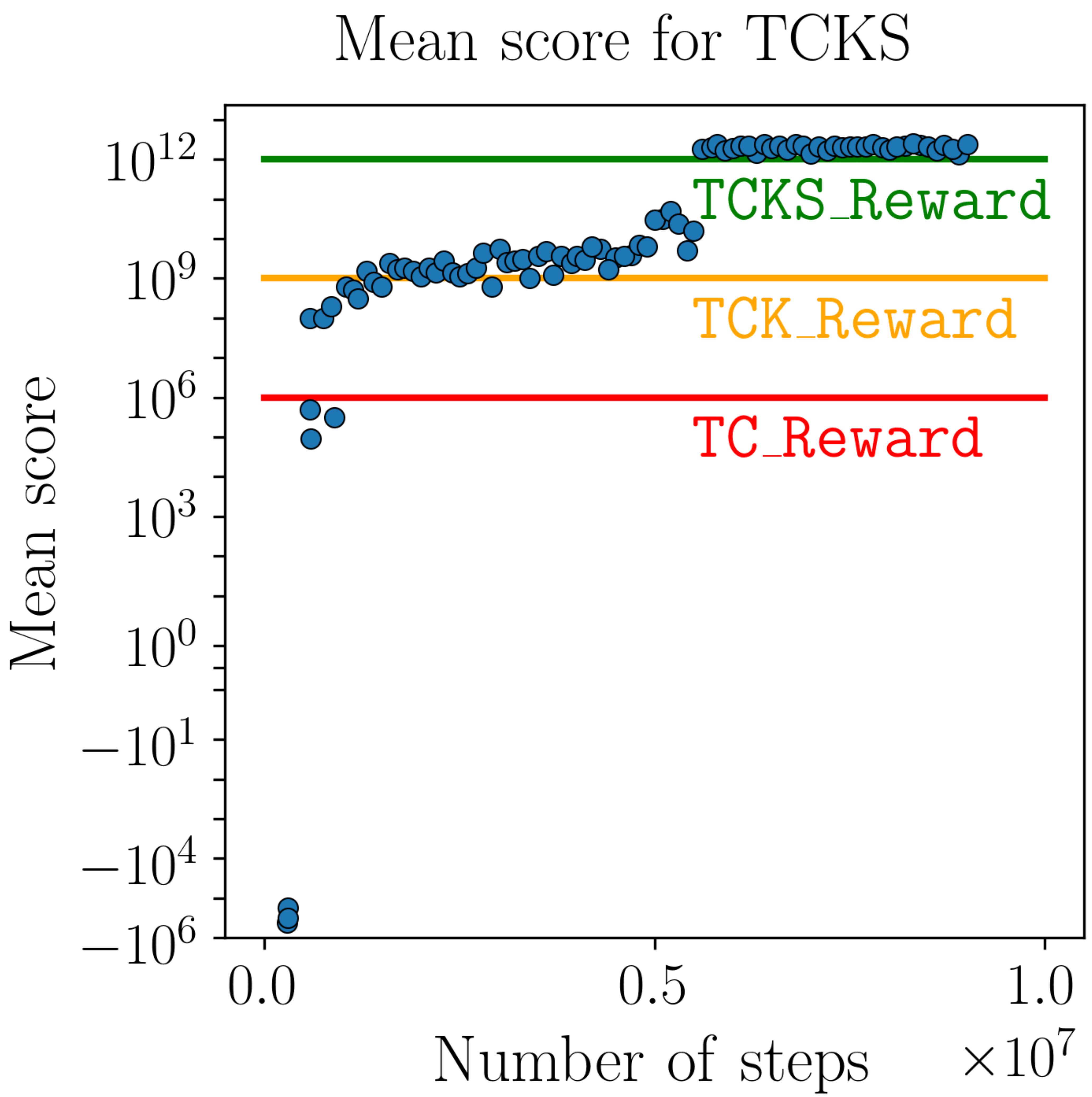

[Halverson, Nelson, FR `19]

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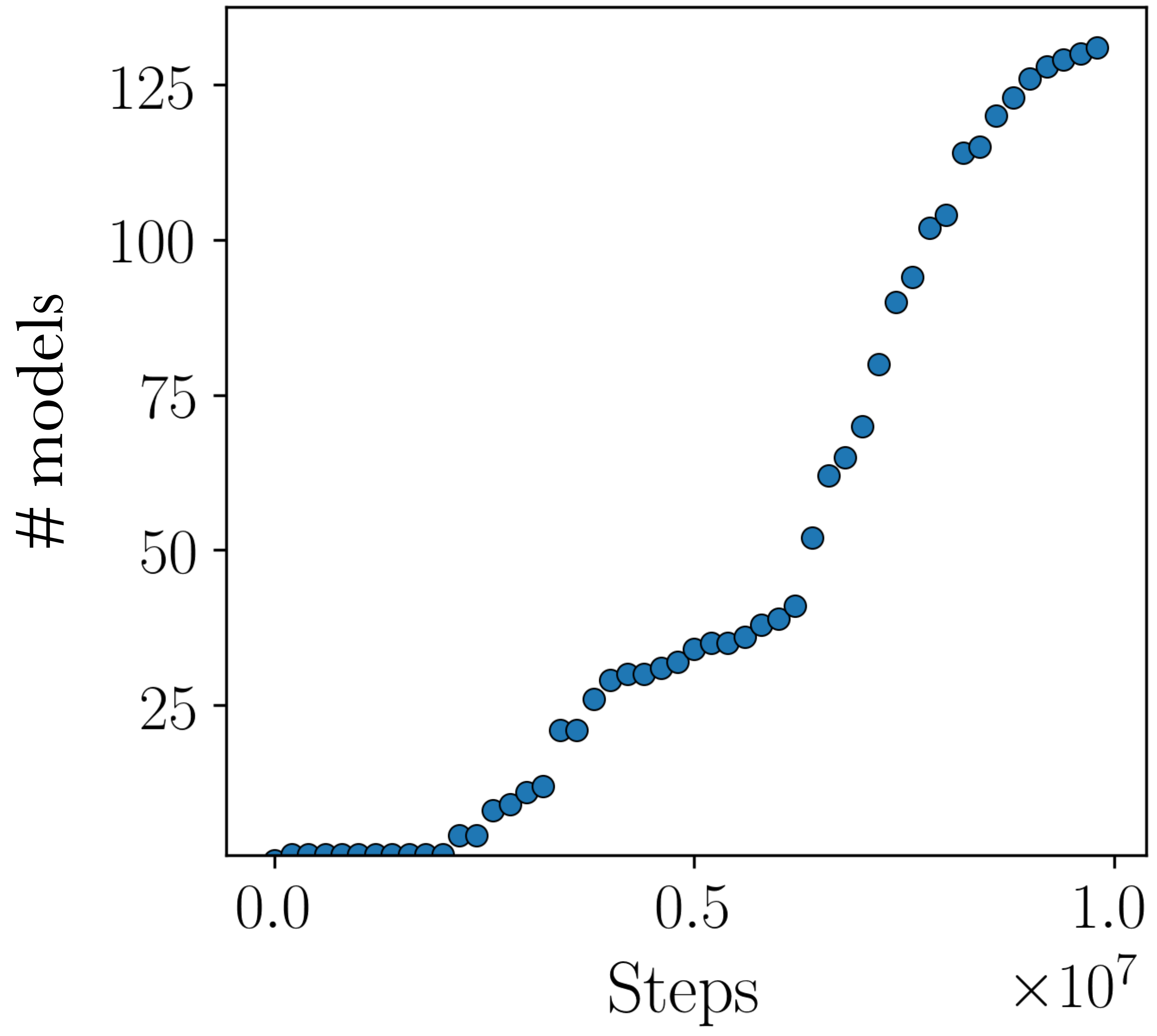
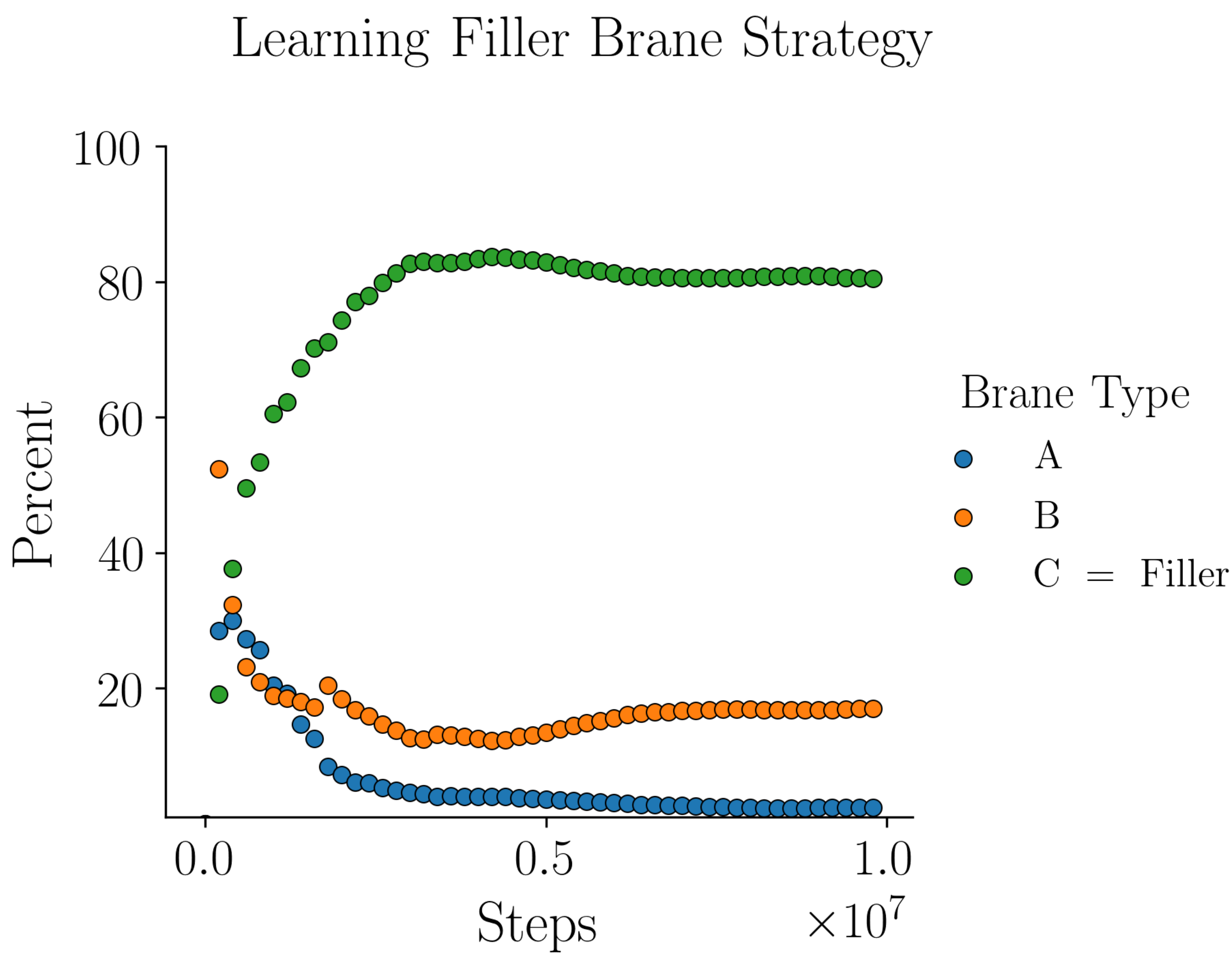
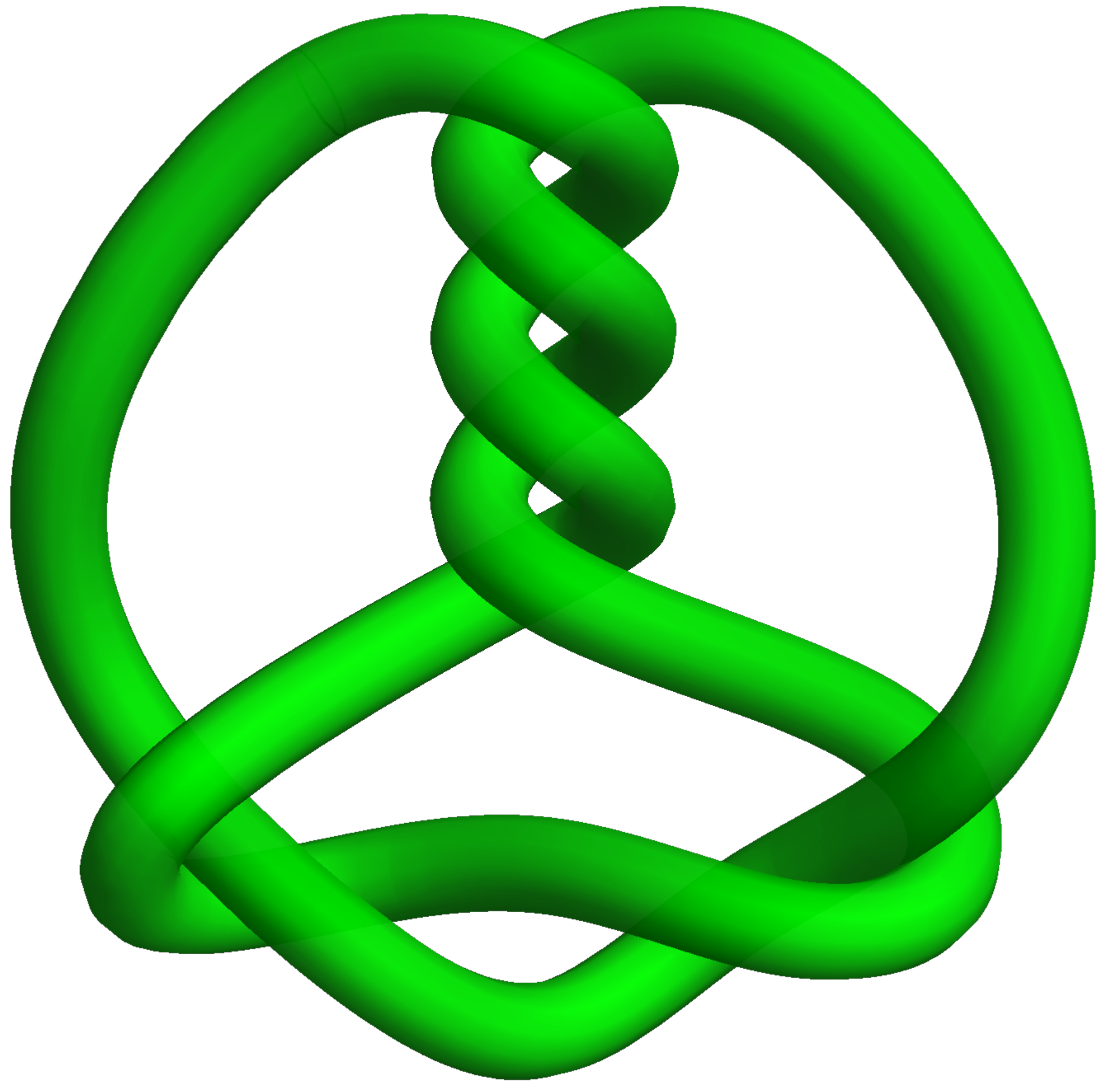
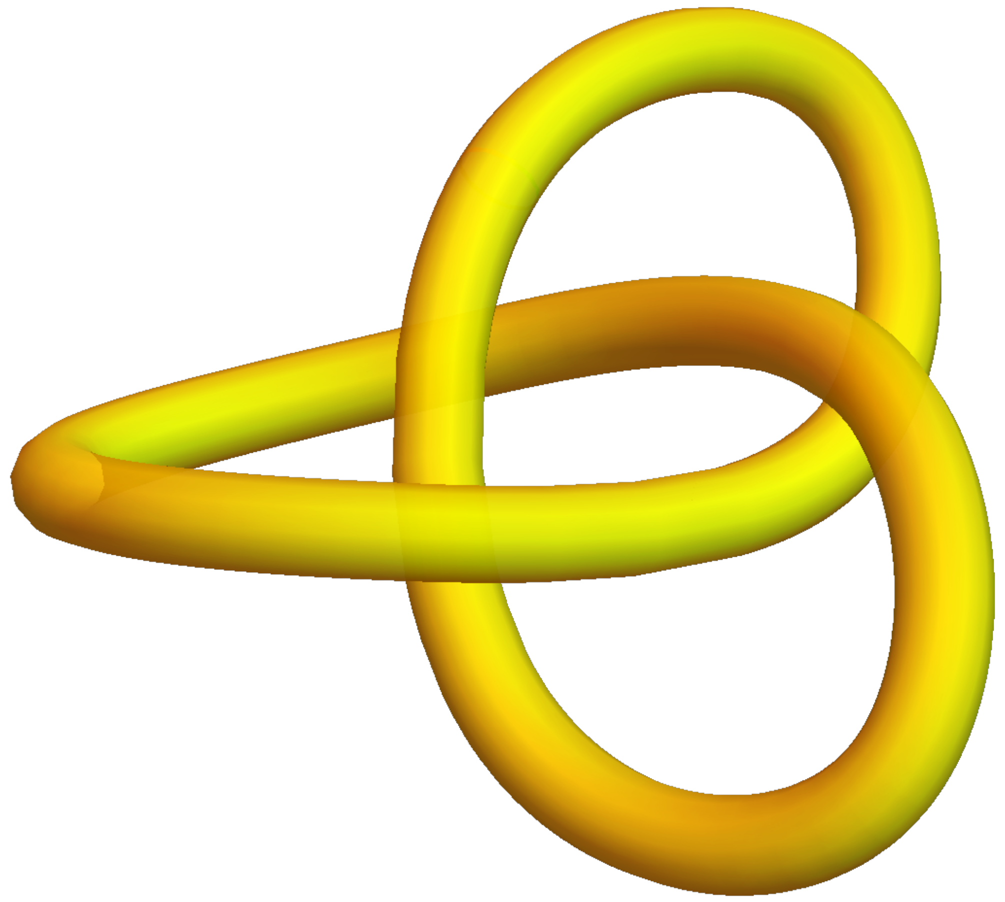
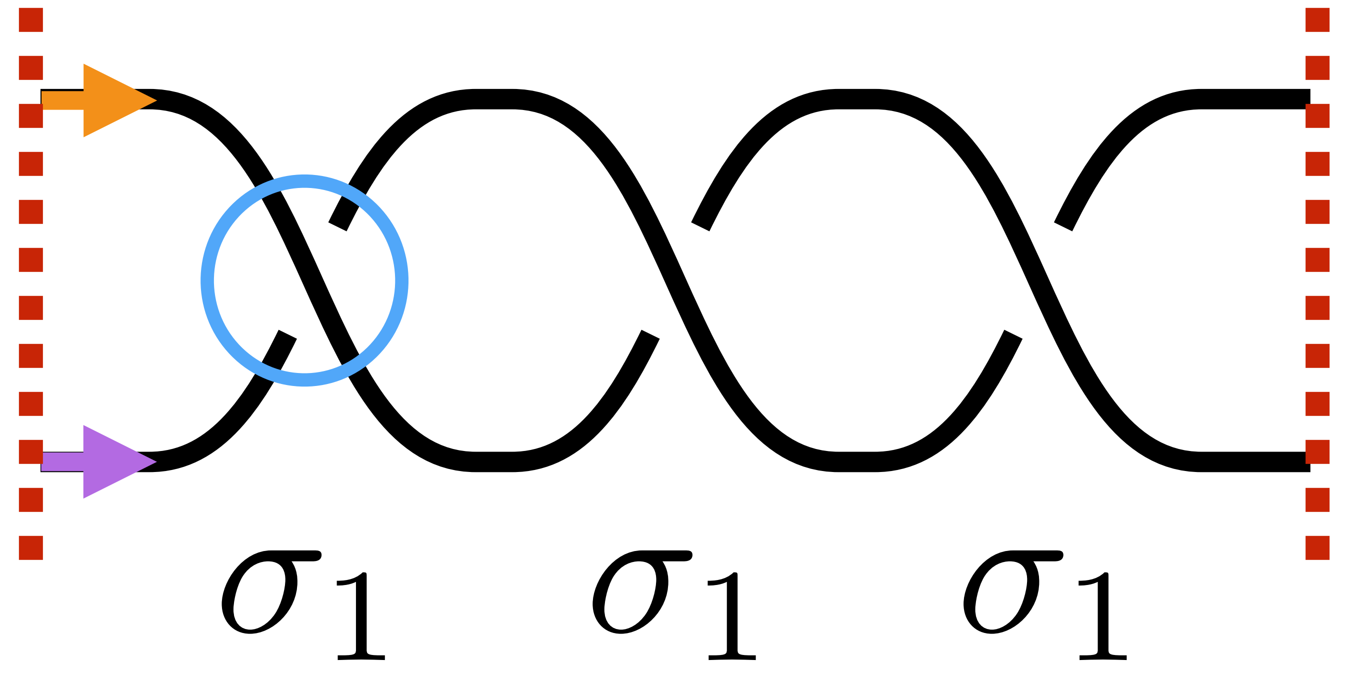




[Halverson, Nelson, FR `19]



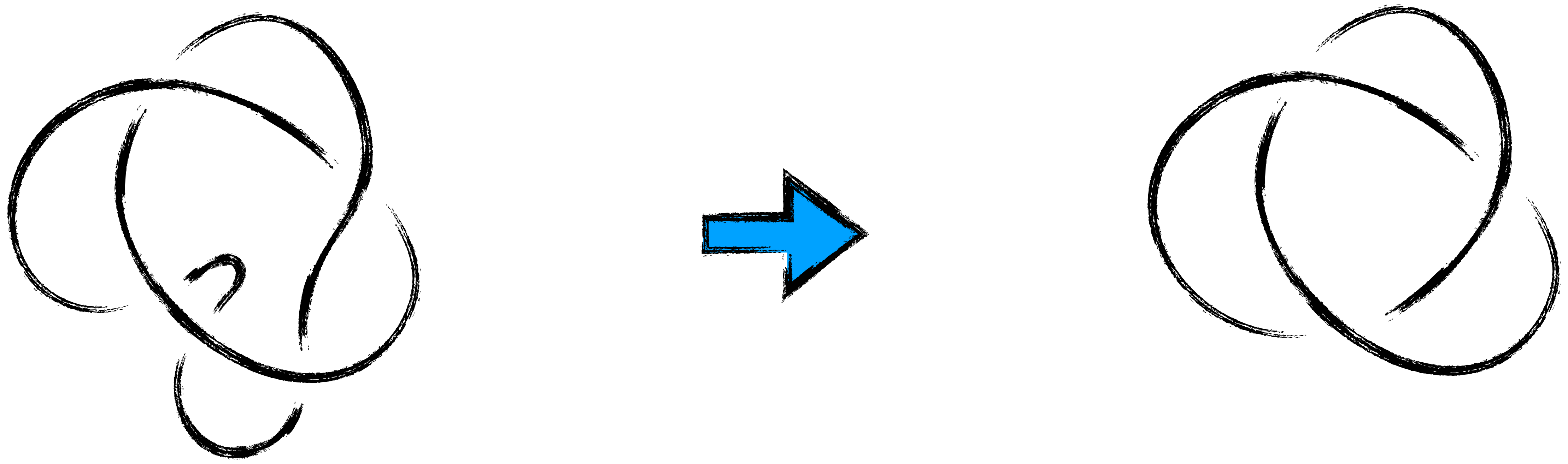




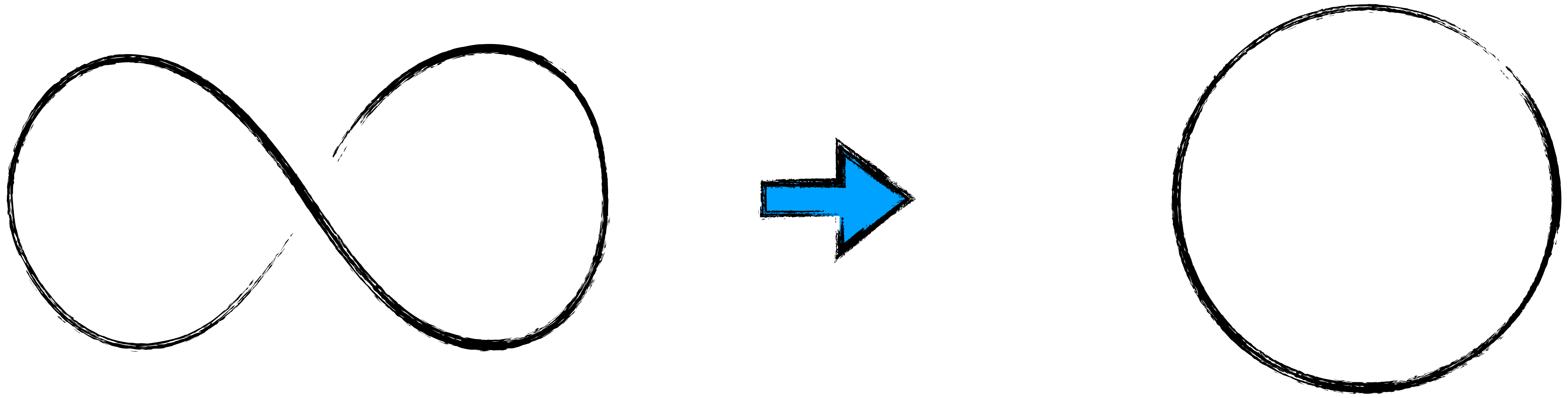
Example II: Knot theory

The unknot problem

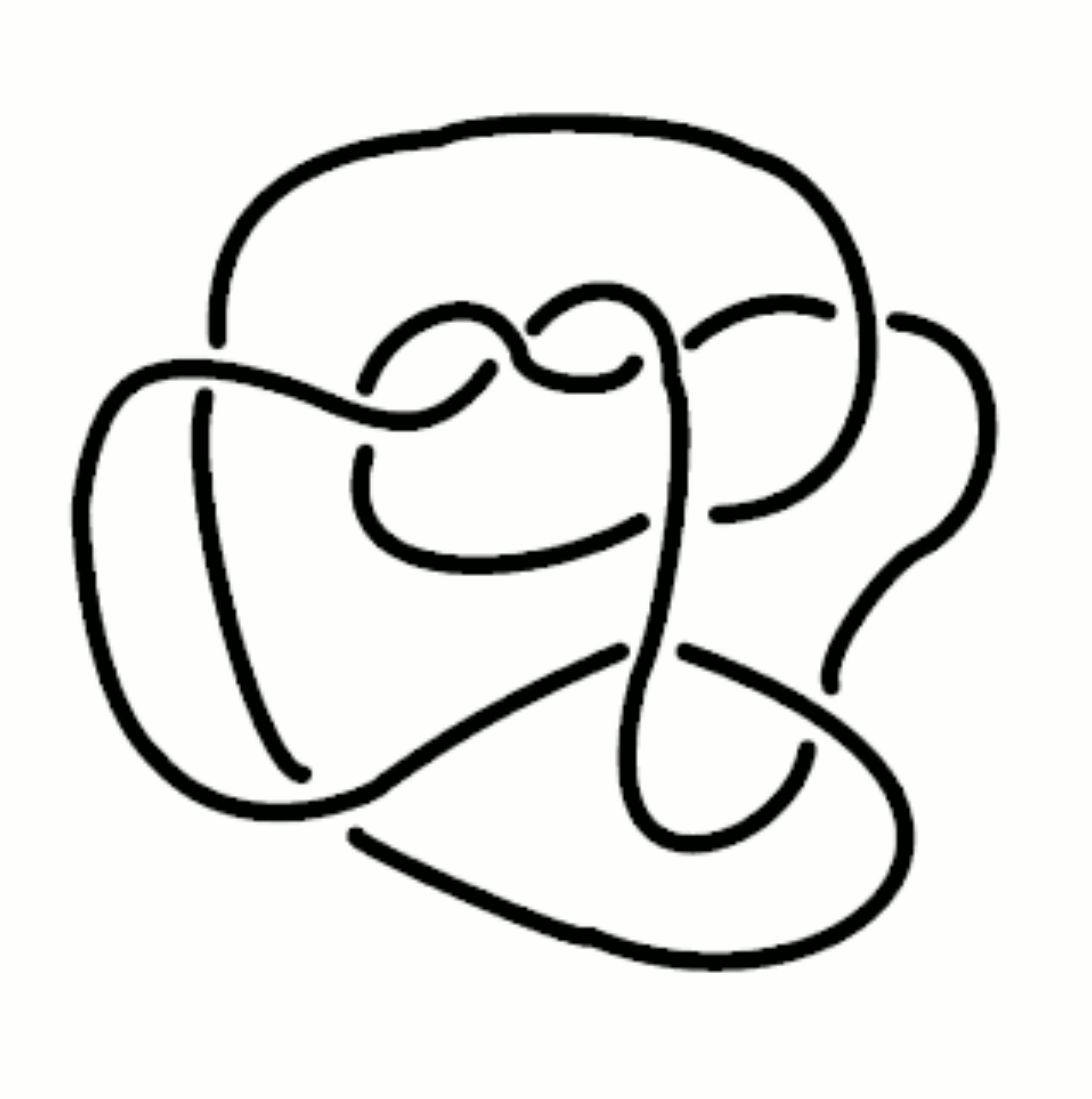
- Simplify a knot as much as possible



- If the knot can be made trivial (i.e. a circle), it is the unknot



Unknot Problem



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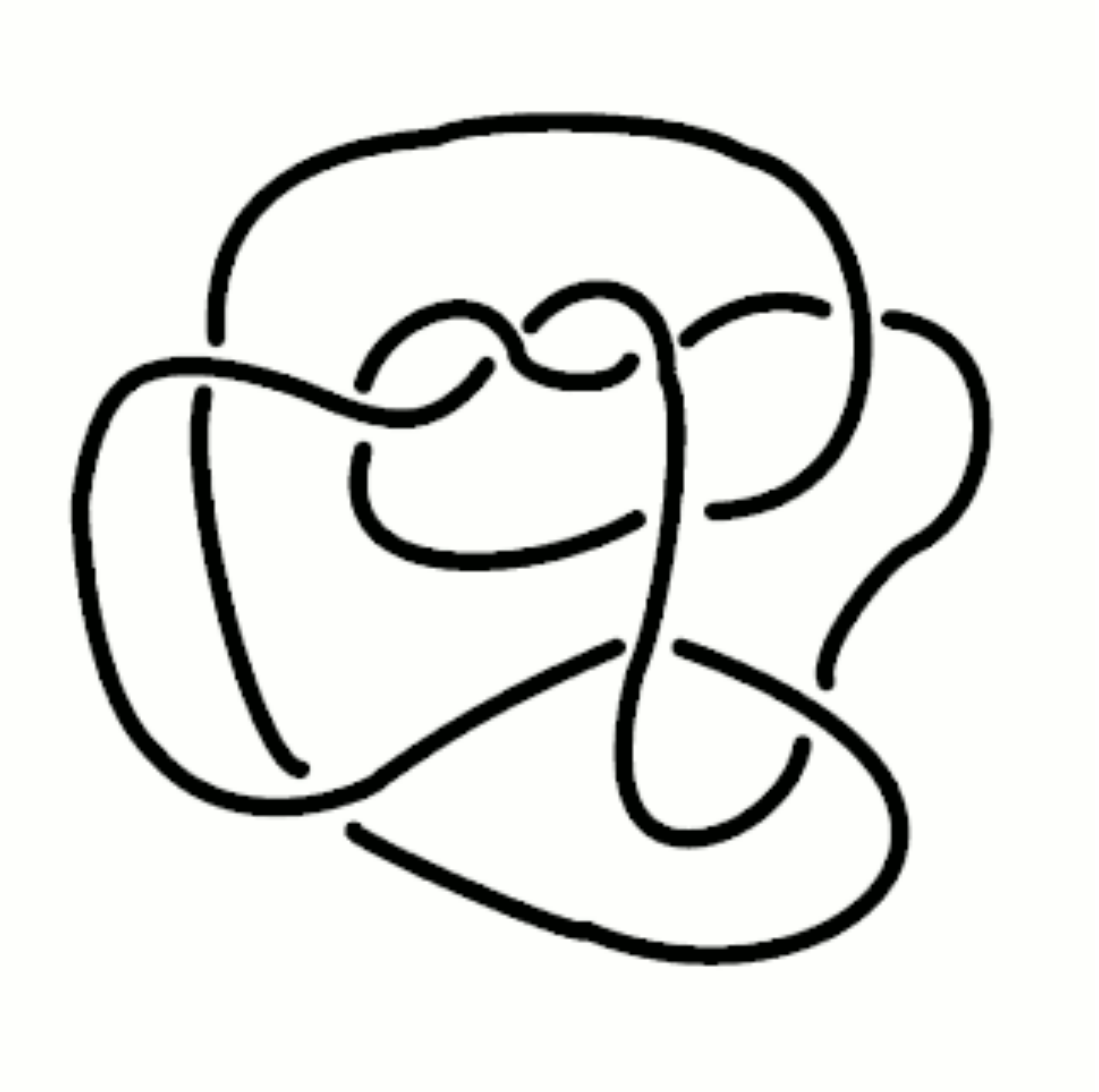
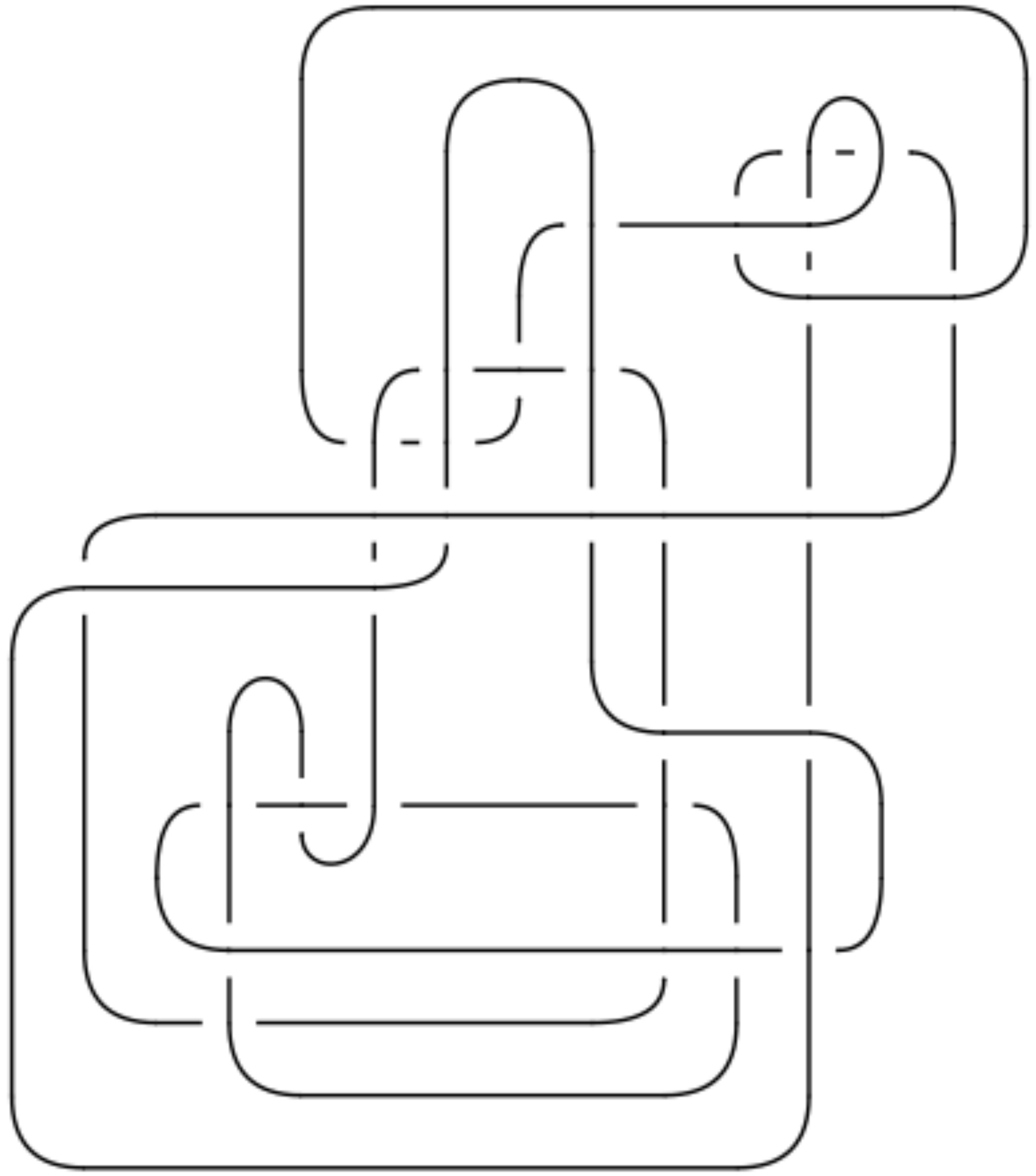
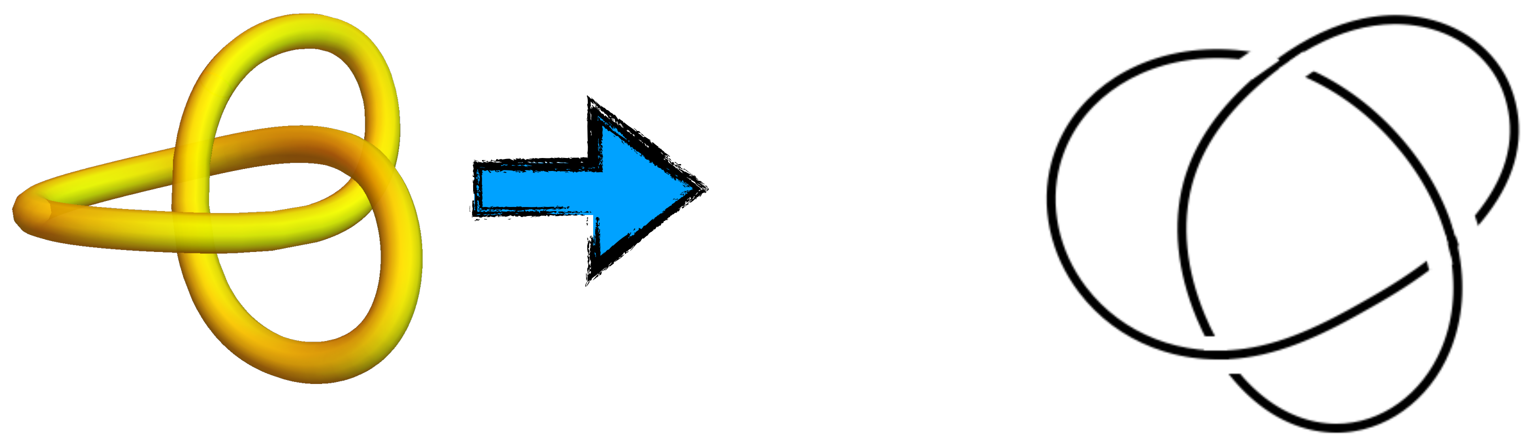




Knot Theory as a natural language problem

- Knots can be represented as words over some alphabet



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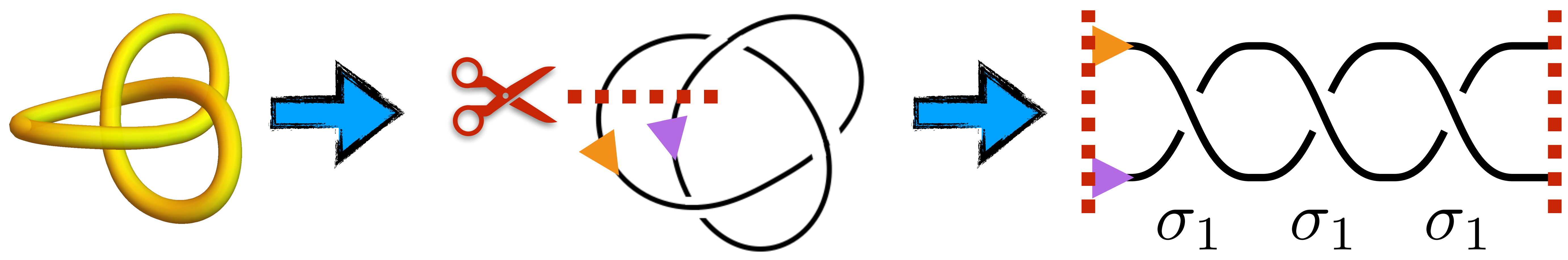

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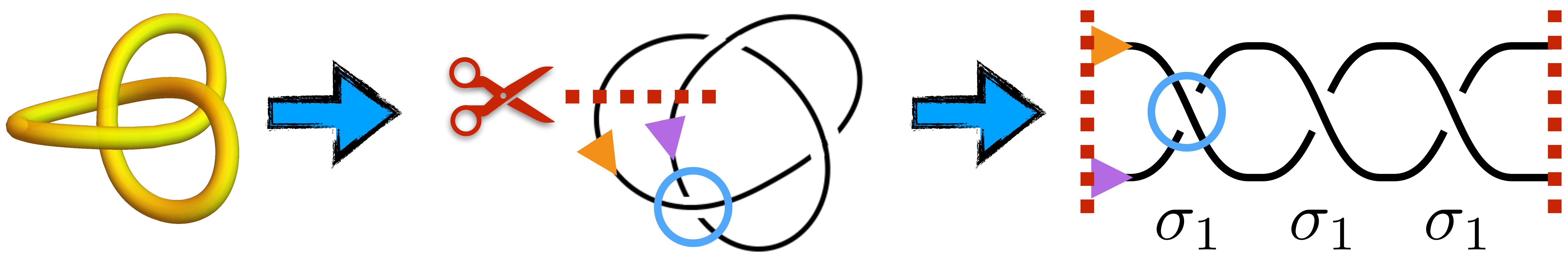
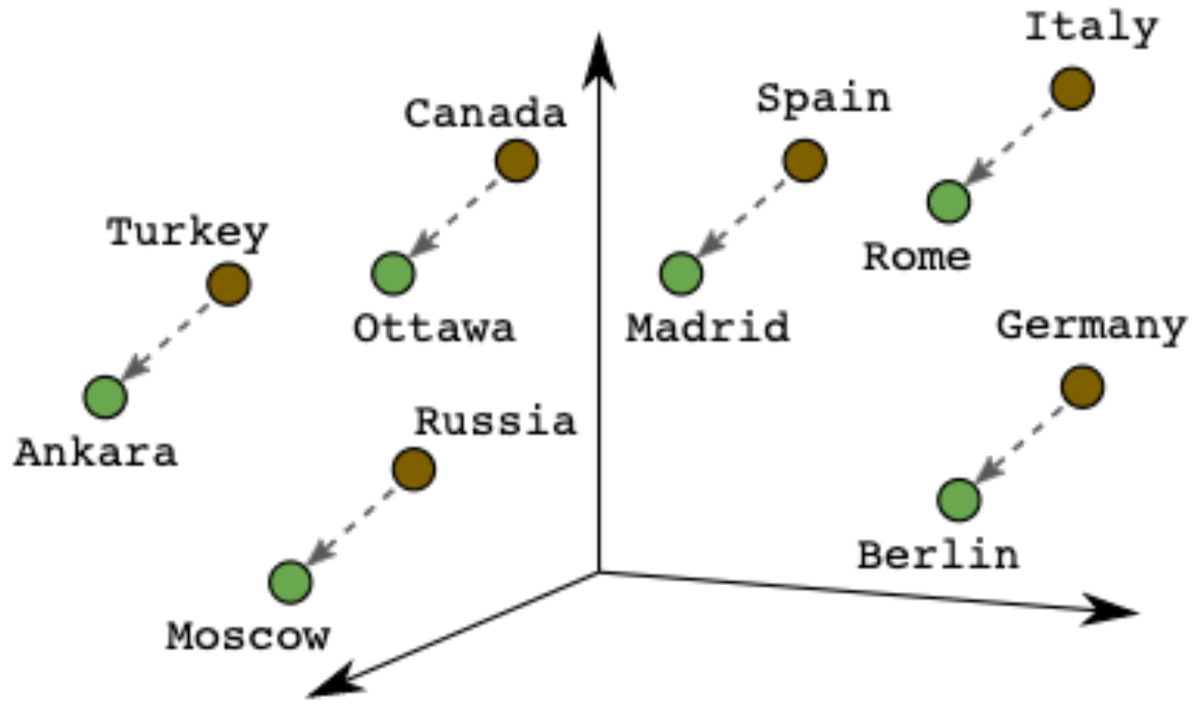


- NLP is something that ML is very good at



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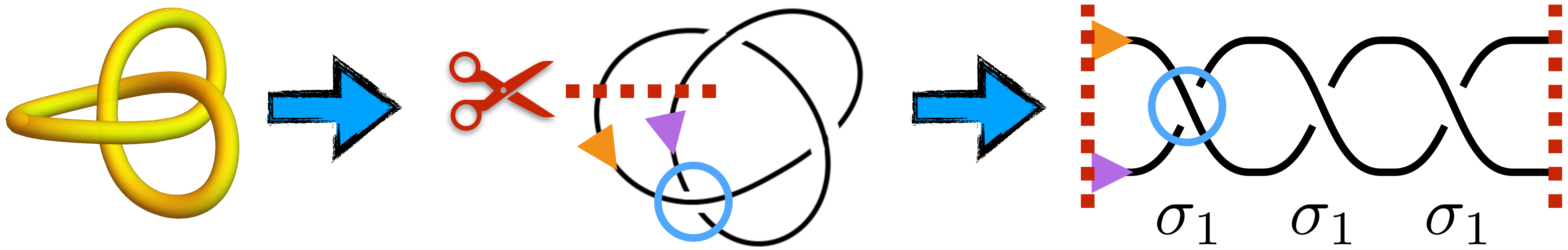

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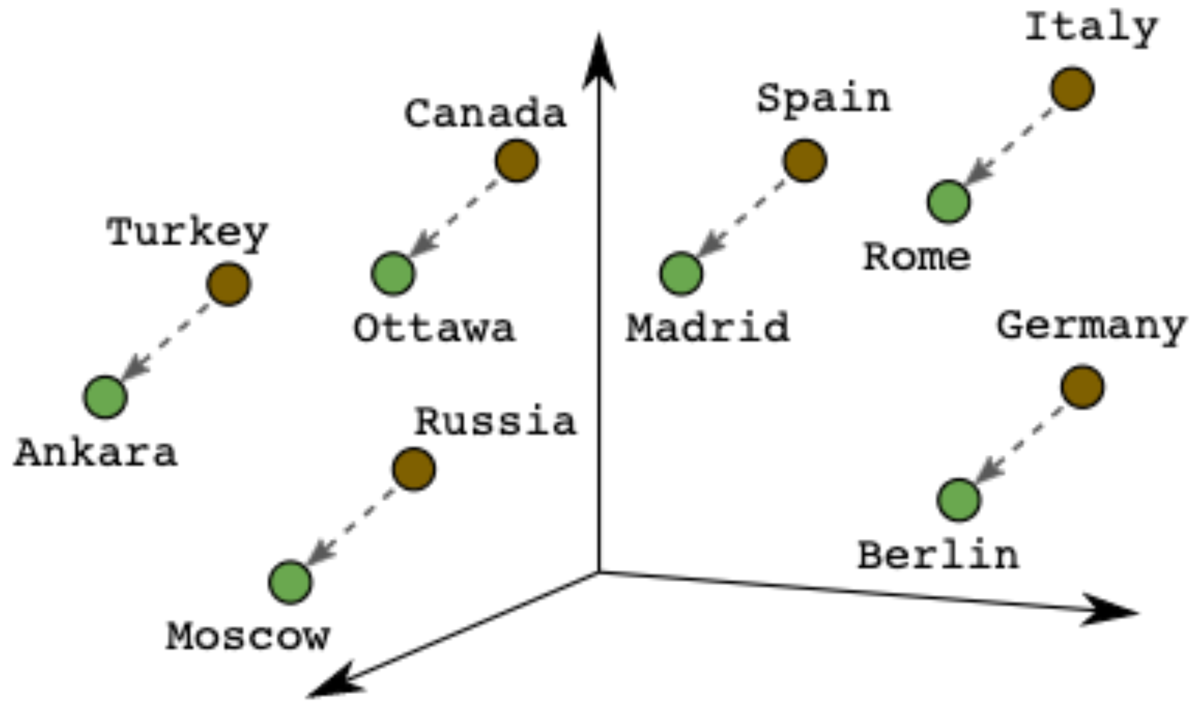
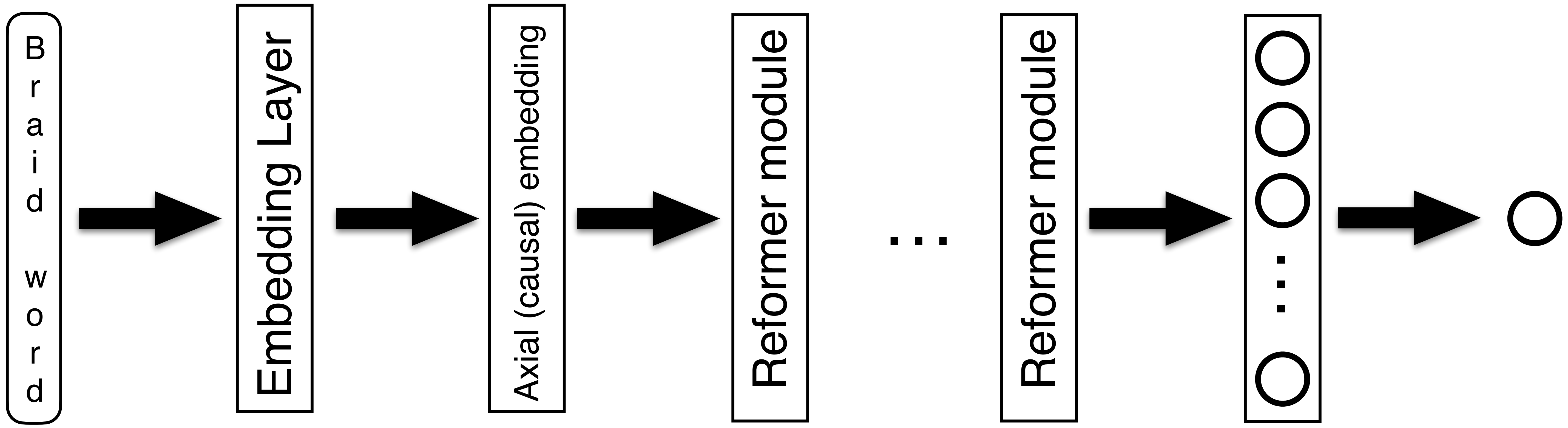
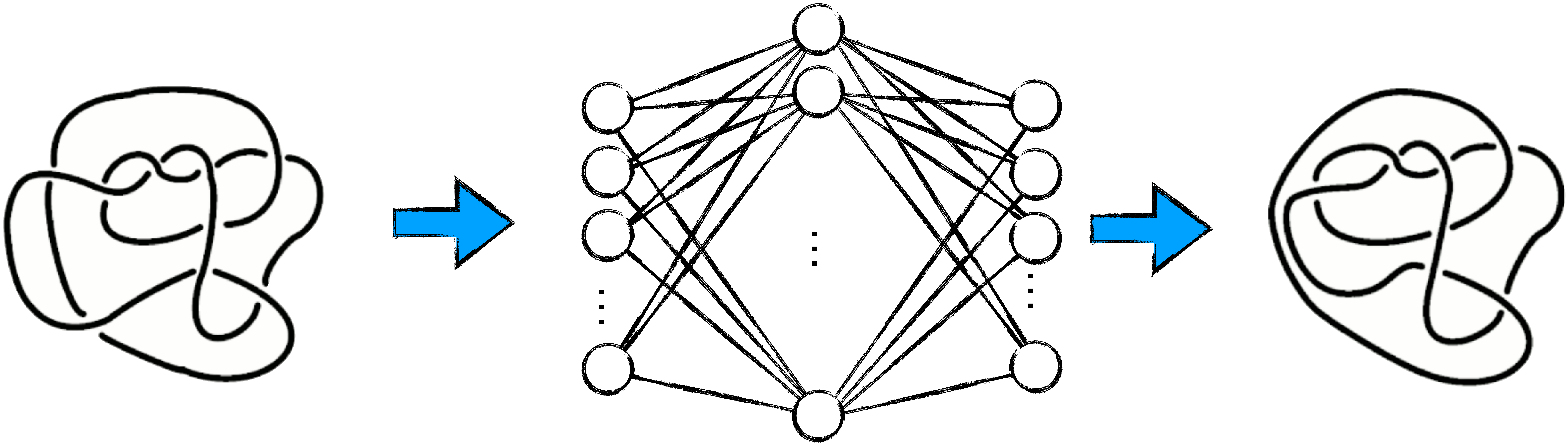


- We use a transformer to tackle the unknot problem



[Vaswani et.al. `17; Kitaev, Kaiser, Levskaya `20]

RL for Knot Theory



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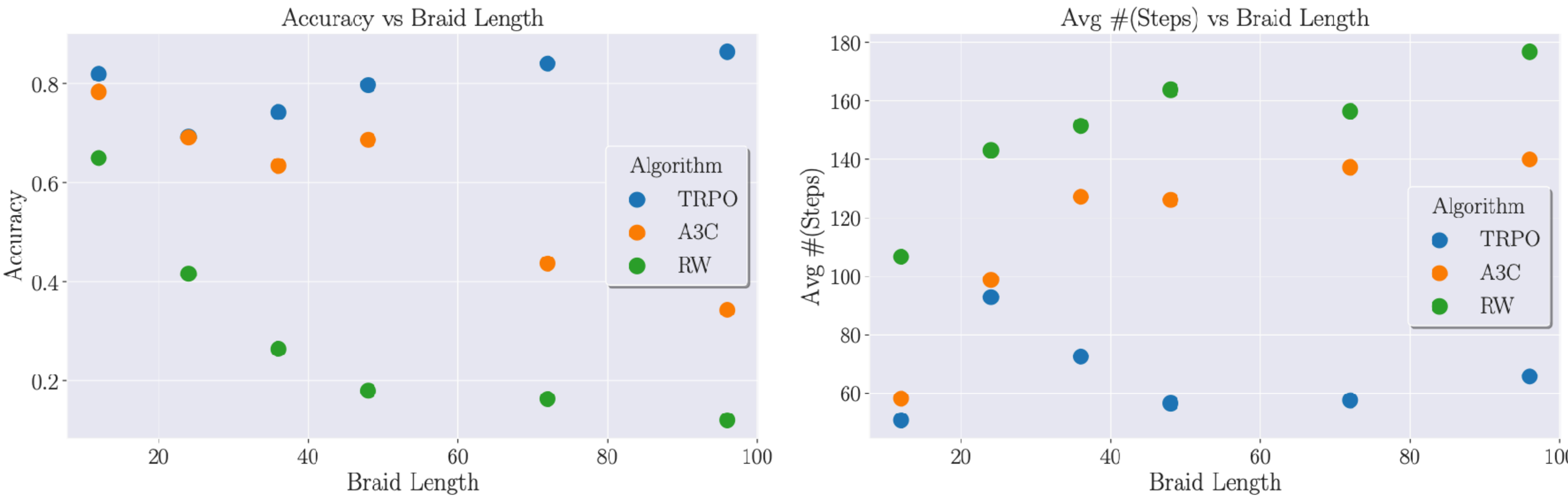


Knot theory

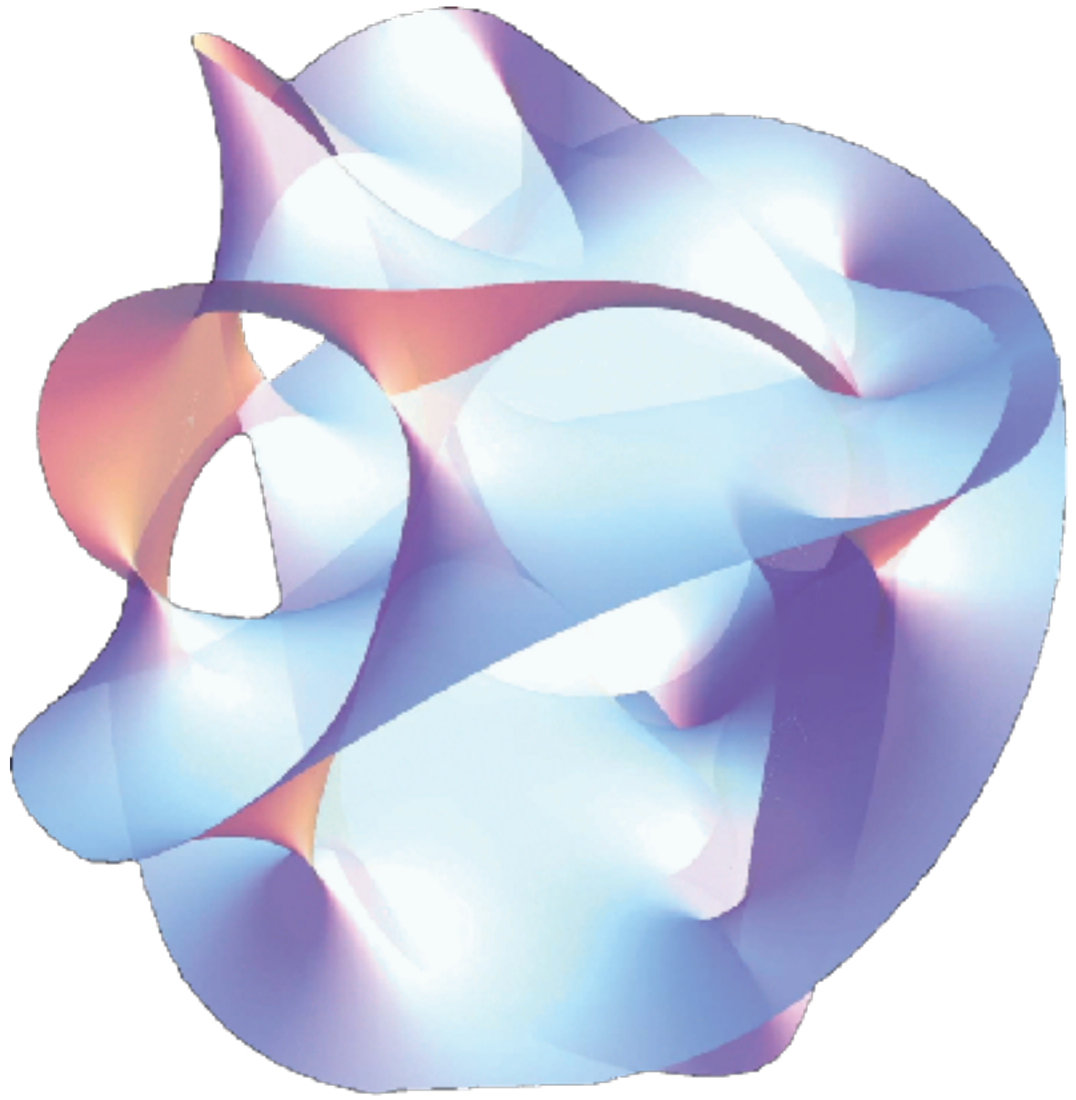
- **Generating Conjectures:**

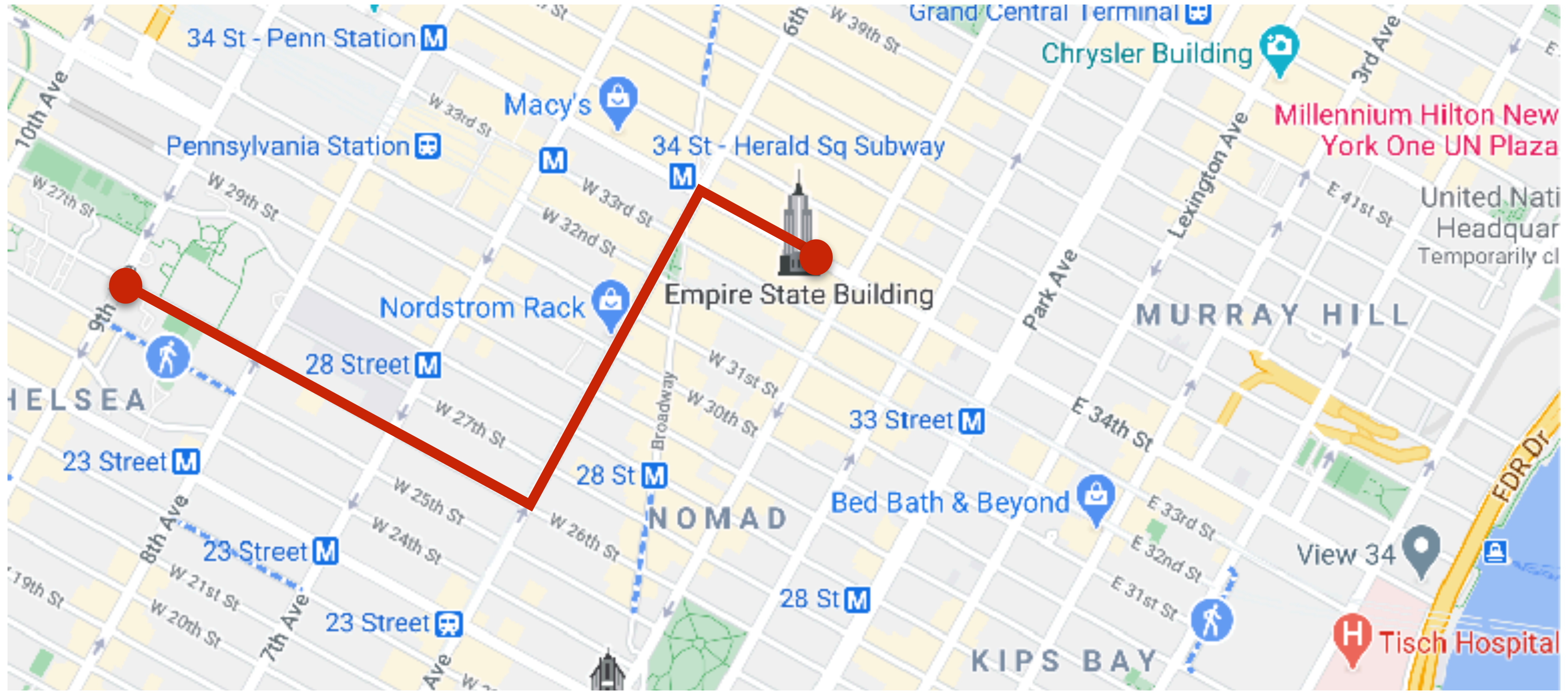
  Train NNs on some knot representation/invariant. If they learn to predict something, hints at a (previously unknown) connection

  - Basic knot invariants $\Leftrightarrow$ quasi-positivity, slice genus, OS τ-invariance [Hughes `16]
  - Jones Polynomial $\rightarrow$ hyperbolic knot volume [Jejjala,Kar,Parrikar `19; Craven,Jejjala,Kar `20]

- **Checking conjectures and mining counter-examples:**

  - sliceness [Hughes `16]
  - smooth Poincare conjecture?



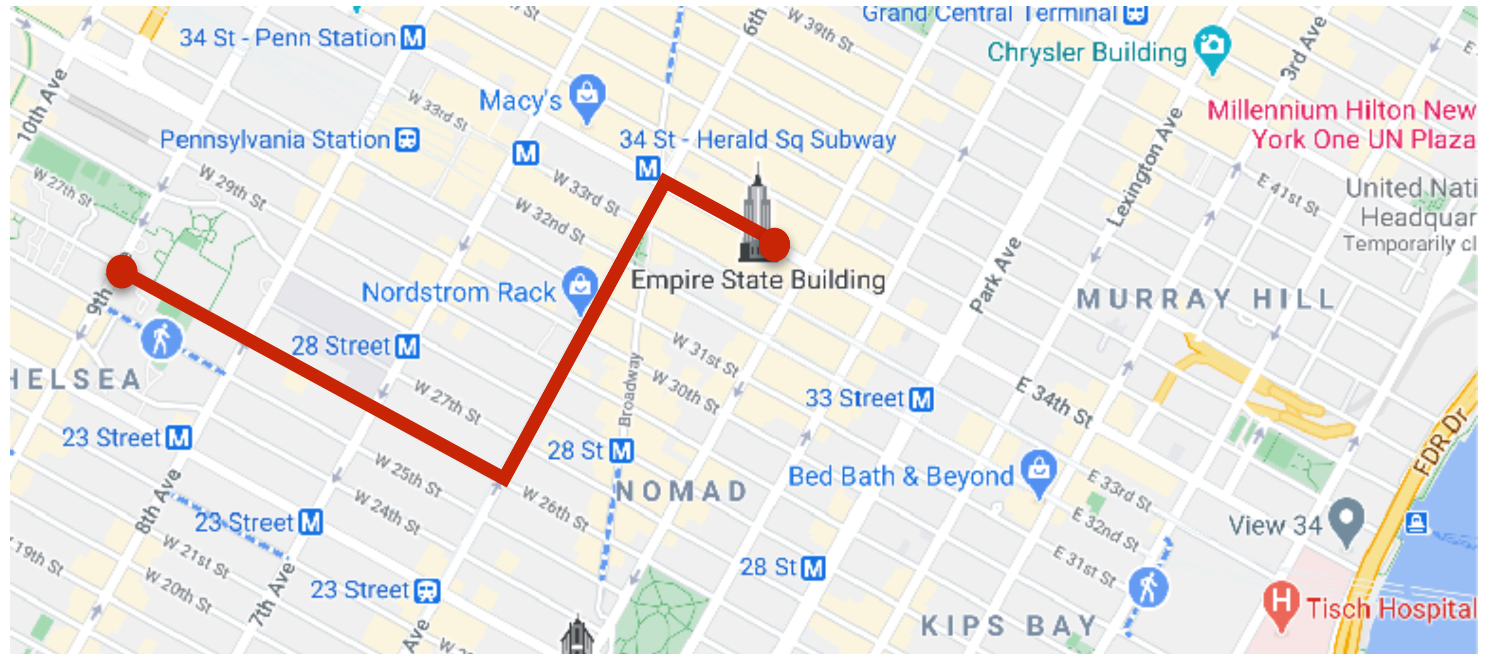


Example III: Calabi-Yau metrics

Metrics

- Metrics measure distances, but the choice is not unique

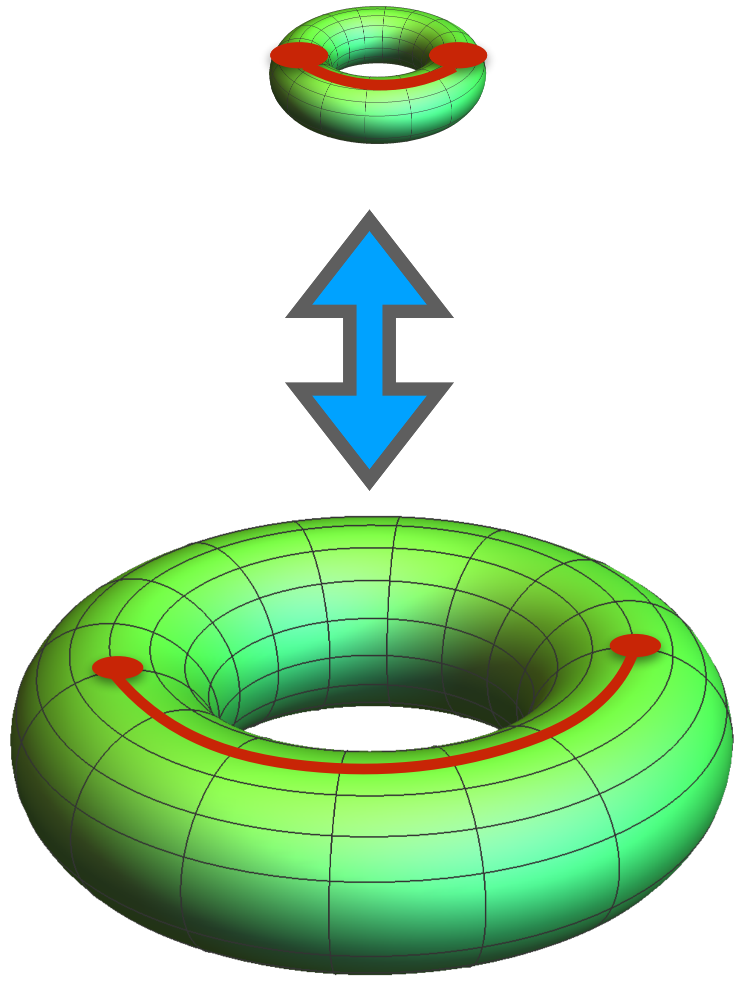
[Source: wikipedia]

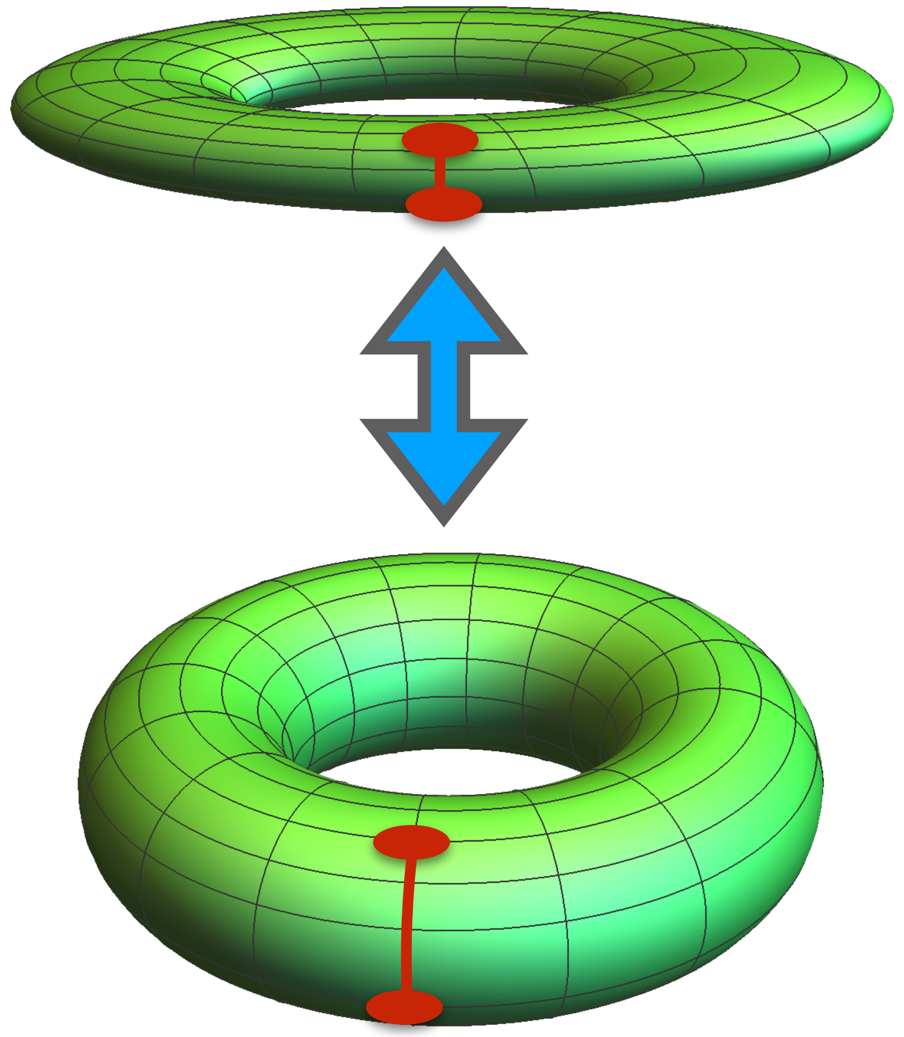


[Source: google maps]

- If space is curved, metric depends on the point you are at. It also depends on volume/shape

[Photo Credit: Jimmy Chin]





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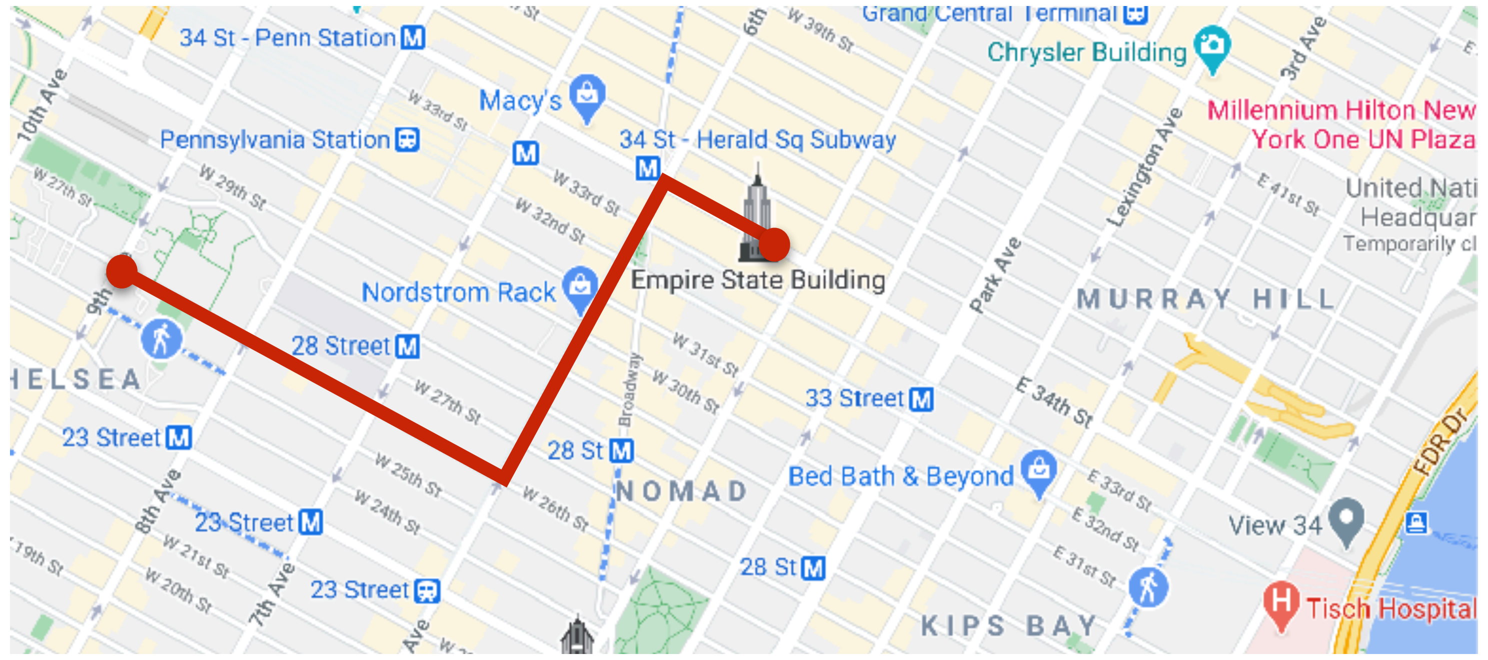


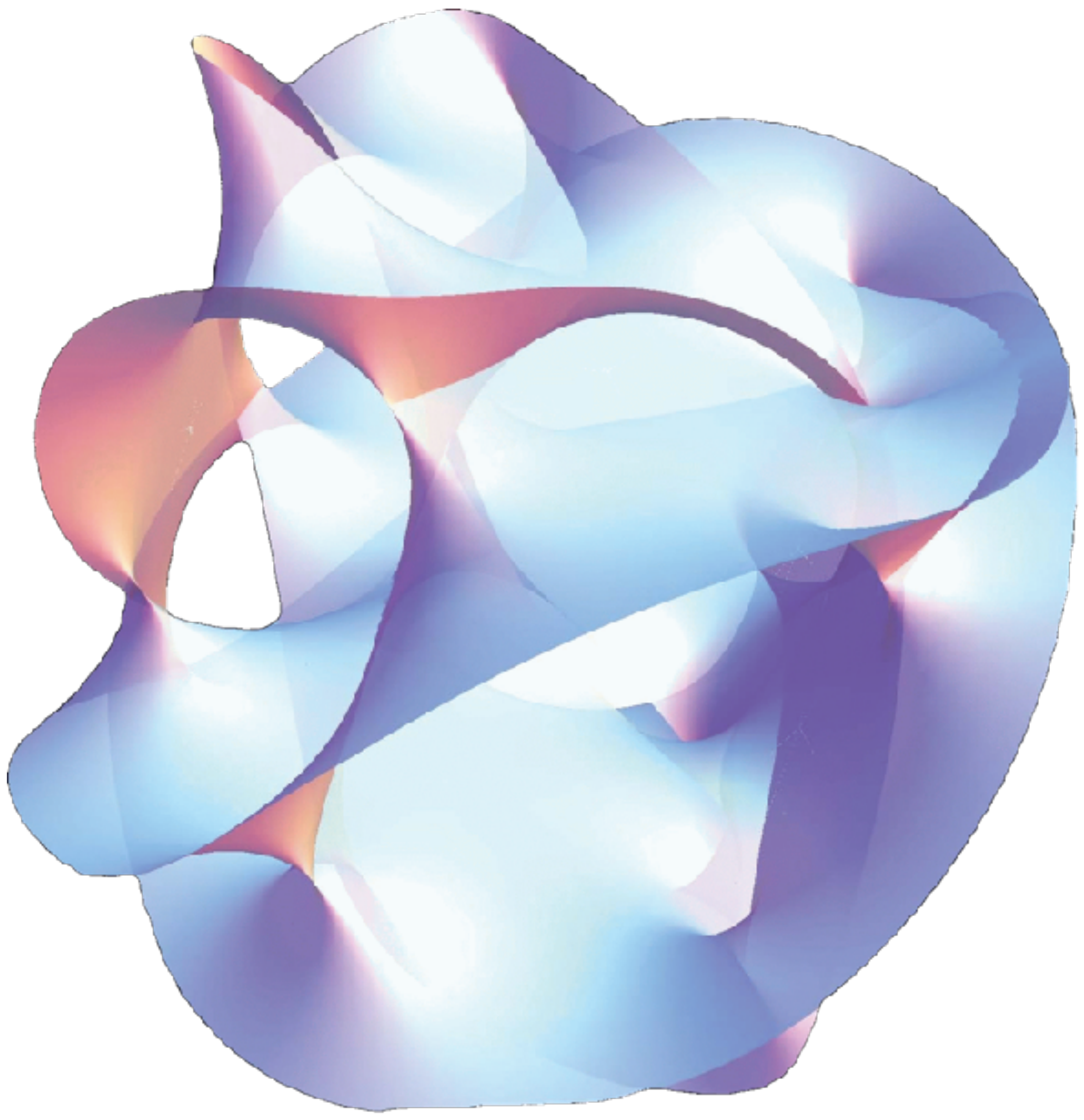
[Source: google maps]

Think of a metric g as a function

$$g: \text{position} \times \text{volume} \times \text{shape} \to \mathbb{R}^{d\times d}$$

and optimize a NN to represent this function subject to the consistency conditions imposed by string theory

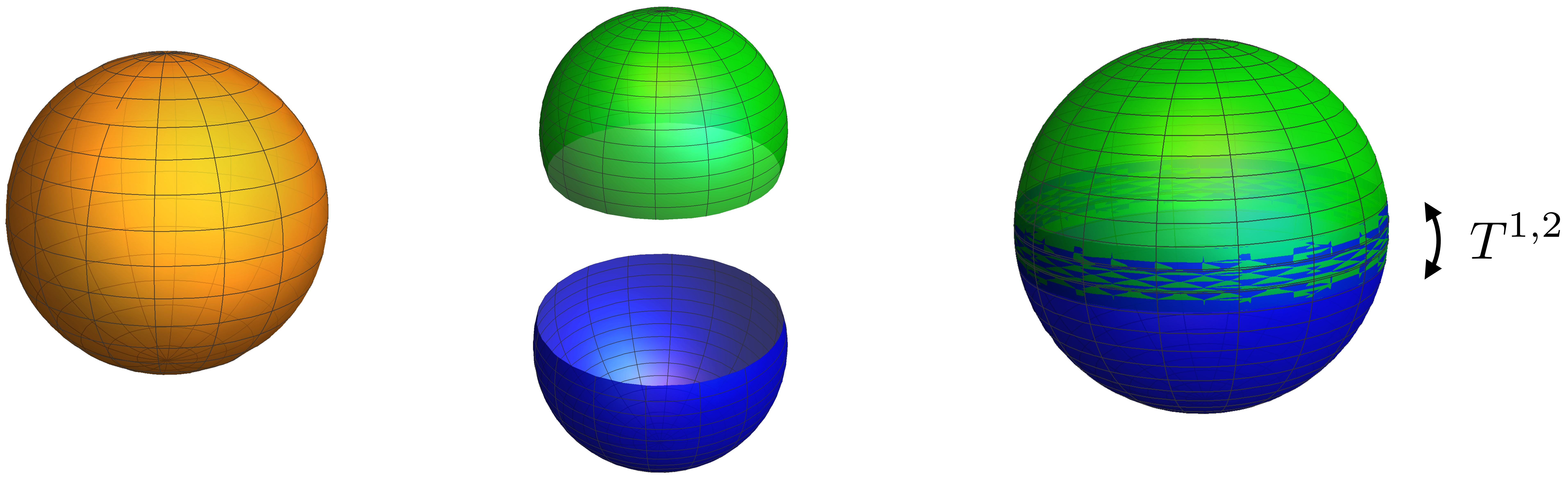
Calabi-Yau manifolds - Properties



- Complex
- Kähler
- Ricci-flat

CY Property 1 - Complex

- In general manifolds cannot be covered by a single patch
- On each patch, one can choose a local description, coordinate system, etc. But one must make sure that the descriptions can be matched on the overlap and everything can be patched to a complex manifold globally (e.g. choice $i=\sqrt{-1}$ vs $i=-\sqrt{-1}$, …)



CY Property 2 - Kähler

- The space must be Kähler
- This means that the metric can be written in terms of derivatives of a real, scalar function called the Kähler potential K

$$g_{a\bar{b}} = \frac{\partial}{\partial z^a}\frac{\partial}{\partial \bar{z}^{\bar{b}}} K \quad , \qquad J = \frac{i}{2}\sum_{a<b} g_{a\bar{b}}\, \varepsilon^{a\bar{b}}\, dz^a d\bar{z}^{\bar{b}} \quad , \quad z = x + iy\,, \quad \bar{z} = x - iy$$

Kähler potential

Kähler form

- In general, integrating the metric to find the Kähler potential is hard. So one can either start with a Kähler potential and derive the metric, or one has to solve the differential equation $\frac{\partial J}{\partial z^a} = \frac{\partial J}{\partial \bar{z}^{\bar{b}}} = 0$.

CY Property 3 - Ricci-flat

- Calabi-Yau spaces are spaces on which a metric exists that is “flat enough”, i.e. their Ricci tensor vanishes

$$
\begin{aligned}
R_{ij} = & -\frac{1}{2}\sum_{a,b=1}^{n}\left(\frac{\partial^2 g_{ij}}{\partial x^a \partial x^b} + \frac{\partial^2 g_{ab}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ib}}{\partial x^j \partial x^a} - \frac{\partial^2 g_{jb}}{\partial x^i \partial x^a}\right) g^{ab} \\
& + \frac{1}{2}\sum_{a,b,c,d=1}^{n}\left(\frac{1}{2}\frac{\partial g_{ac}}{\partial x^i}\frac{\partial g_{bd}}{\partial x^j} + \frac{\partial g_{ic}}{\partial x^a}\frac{\partial g_{jd}}{\partial x^b} - \frac{\partial g_{ic}}{\partial x^a}\frac{\partial g_{jb}}{\partial x^d}\right) g^{ab} g^{cd} \\
& - \frac{1}{4}\sum_{a,b,c,d=1}^{n}\left(\frac{\partial g_{jc}}{\partial x^i} + \frac{\partial g_{ic}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^c}\right)\left(2\frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d}\right) g^{ab} g^{cd} \\
= 0
\end{aligned}
$$

- Note that ensuring g is Kähler introduces 2 more derivatives since $g_{a\bar{b}} = \frac{\partial}{\partial z^a}\frac{\partial}{\partial \bar{z}^{\bar{b}}} K$

CY Property 3 - Ricci-flat

- This fourth-order partial differential equation is extremely hard to solve
- We can improve on this. On a CY, one can write down

$$J = \frac{i}{2} \sum_{a<b} g_{a\bar{b}} \, \varepsilon^{a\bar{b}} \, dz^a d\bar{z}^{\bar{b}} \qquad \Rightarrow \qquad J^3 = -\frac{i}{8} \sqrt{\det g} \, dz_1 \, d\bar{z}_1 \, dz_2 \, d\bar{z}_2 \, dz_3 \, d\bar{z}_3$$

$$\Omega = \left(\frac{\partial p}{\partial z_4} \right)^{-1} dz_1 \, dz_2 \, dz_3 \quad \Rightarrow \quad |\Omega|^2 = \left| \frac{\partial p}{\partial z_4} \right|^{-2} dz_1 \, dz_2 \, dz_3 \, d\bar{z}_1 \, d\bar{z}_2 \, d\bar{z}_3$$

- Since the volume form is unique (up to a constant):

$$J^3 = \kappa |\Omega|^2$$

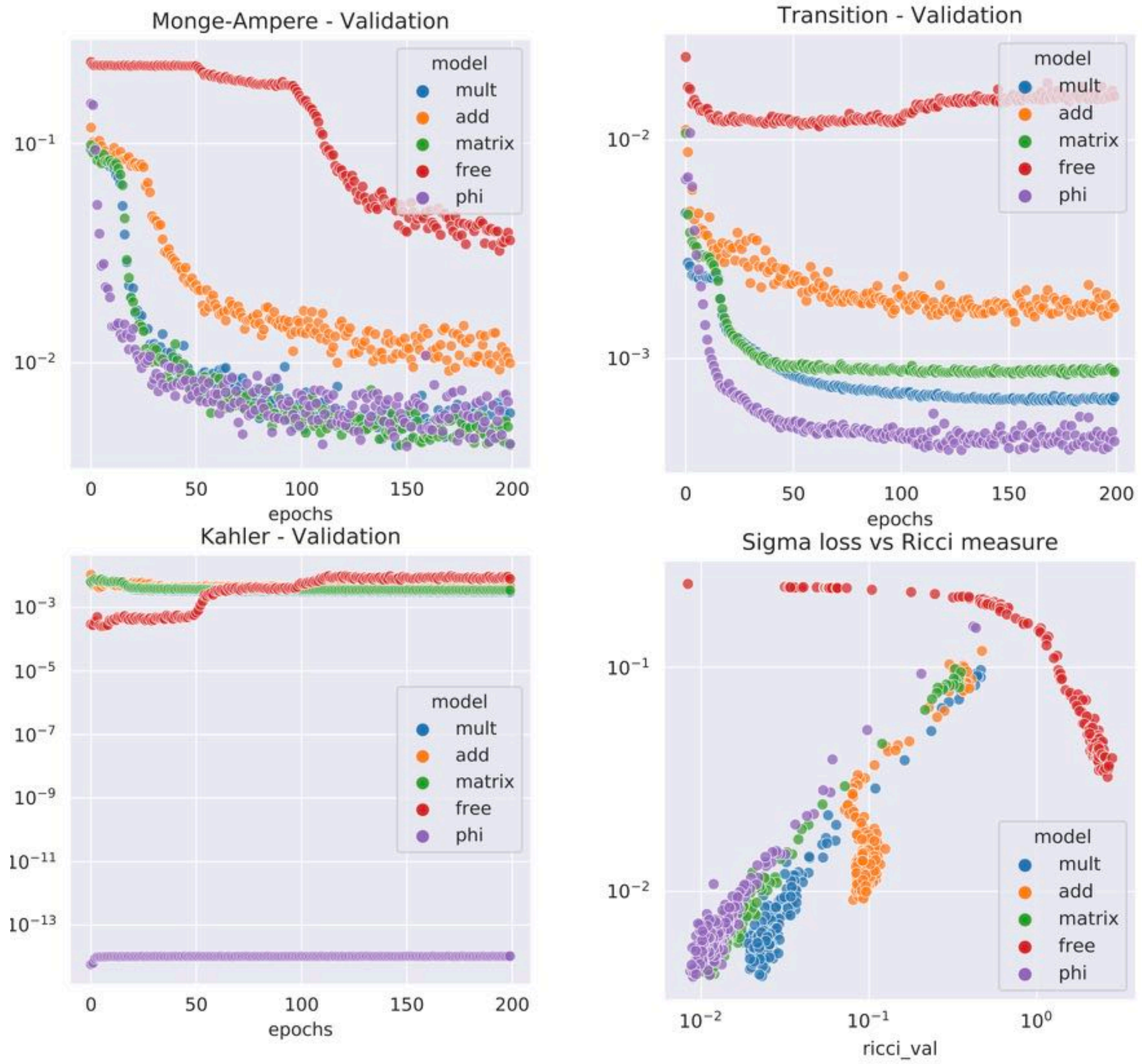
CY metric ansatze

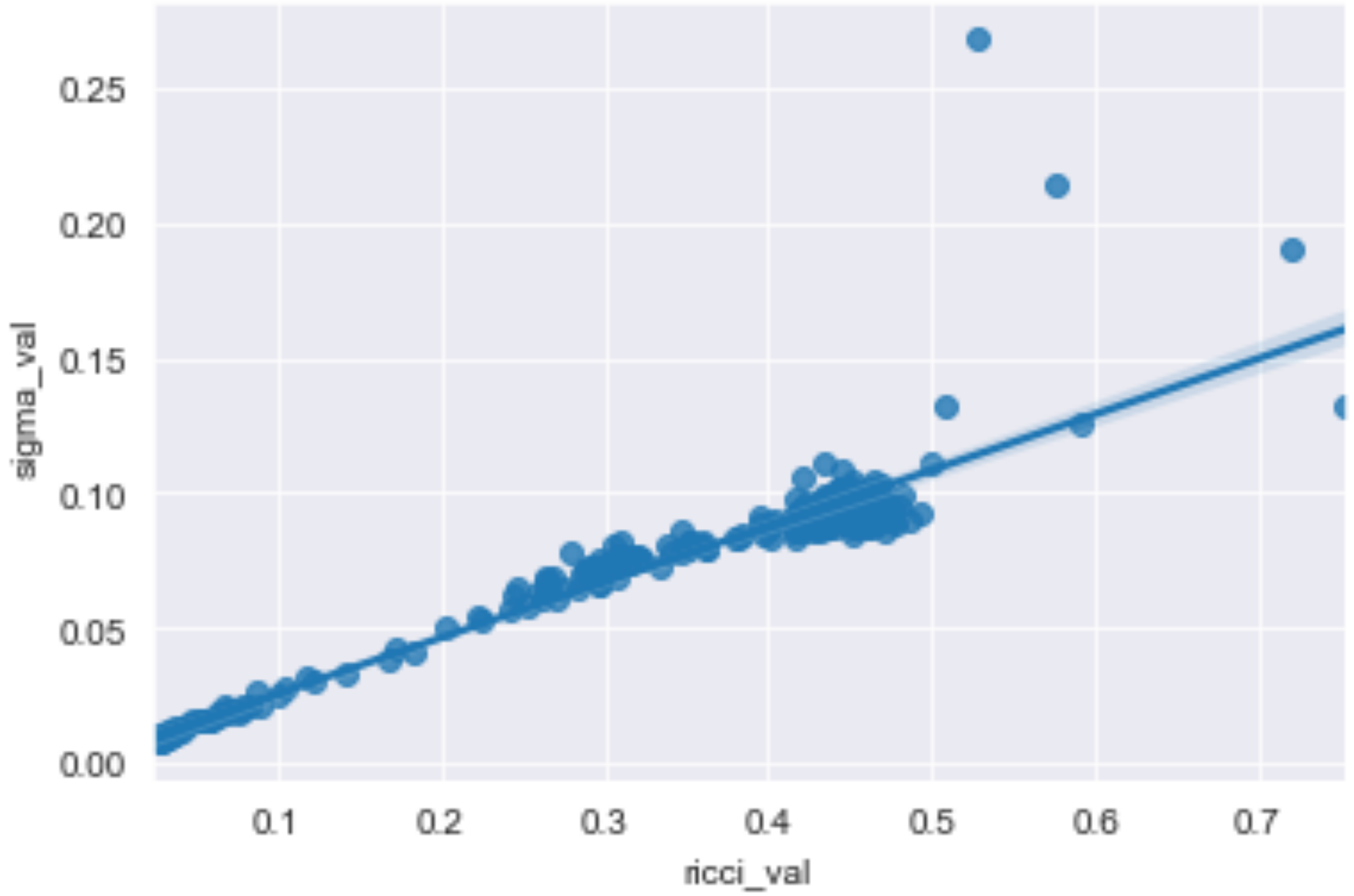
- The condition $J^3 = \kappa|\Omega|^2$ can be turned into a (Monge-Ampere) PDE
- As it turns out, we can ensure the complex and Kahler property and keep the volume moduli fixed if we write

$$g_{\rm CY} = g_{\rm reference} + \partial\bar{\partial}\Phi$$

  and approximate the (scalar) function $\Phi = \Phi(\text{position}, \text{shape})$ with a NN
- Other possibilities (can depart from Kahler and fixed volume):
  - $g_{\rm CY} = g_{\rm NN}$ (works the least well)
  - $g_{\rm CY} = g_{\rm reference} + g_{\rm NN}$ (works better)
  - $g_{\rm CY} = g_{\rm reference}(\mathbb{1} + g_{\rm NN})$ (works best; as well as the $\partial\bar{\partial}\Phi$ approach)

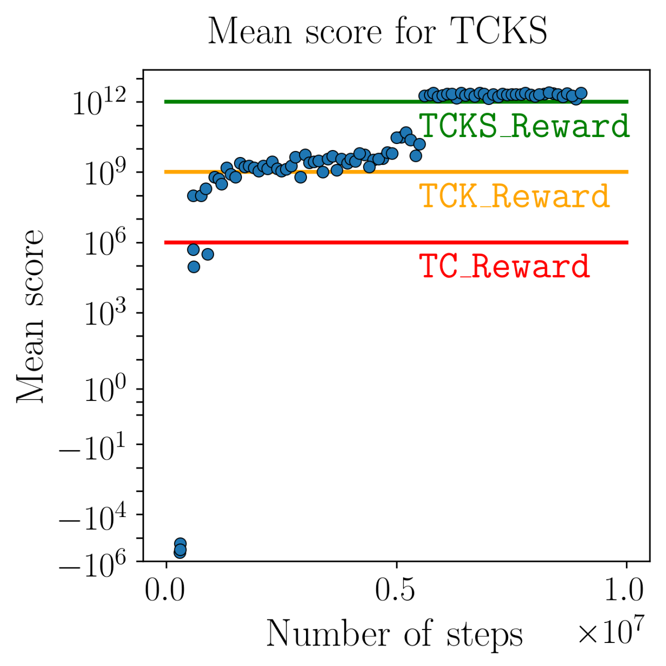
CY metric results

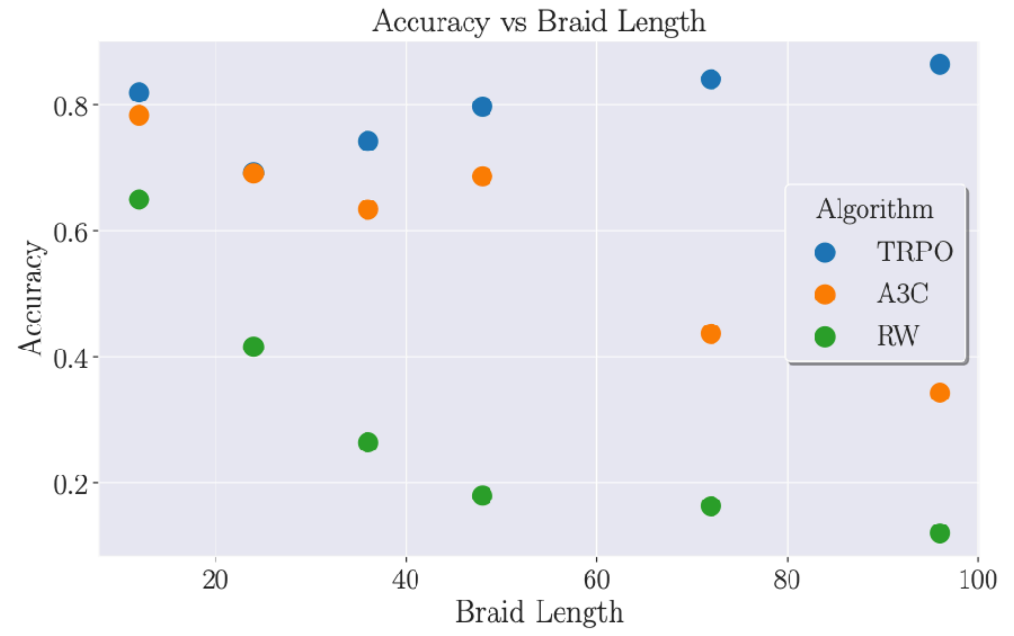


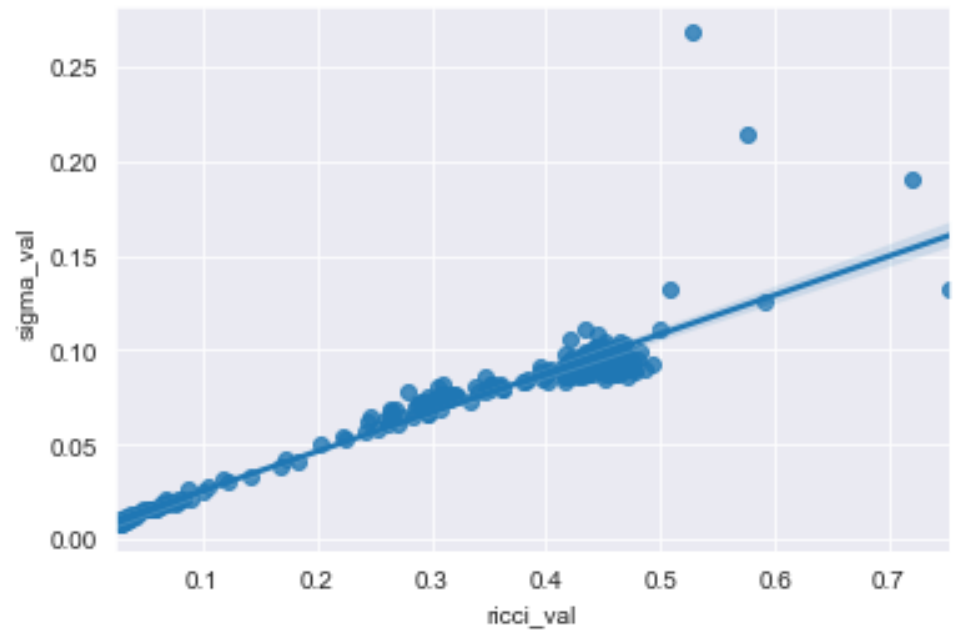


Conclusions

- String theory comes with discrete, hard combinatorial problems that seem amendable to RL
  - Solve Diophantine equations
  - Find the unknot
- ML techniques from other areas can be imported and successfully applied
  - Mapping knot theory to NLP
- String theory's continuous problems can be solved with fast optimization
  - PDE for CY metrics







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Thank you for your attention!