Learning Reasoning Components

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Introduction We describe two iterative algorithms that combine high-level proof state evaluation and strategic reasoning decisions with guided low-level saturation-style proof search. For each part, we learn the tasks using an efficient logic-aware graph neural network [14] (GNN) recently integrated [8] into the ENIGMA [9, 4] guidance system of E [16, 17]. The general motivation is to explore and develop more human-like reasoning architectures, i.e., systems combining various (Malarious [20]) AI components and learning/reasoning feedback loops, which are (preferably) also competitive with ATPs in resource-controlled large-theory settings.[1]

GNN-ENIGMA The novelty previously introduced by GNN-ENIGMA compared to other saturation provers is that the generated clauses are not ranked immediately and independently of other clauses. Instead, they are judged by the GNN in larger batches and with respect to a large number of already selected clauses – the *context*. The GNN estimates collectively the most useful subset of the context and new clauses (*queries*) by several rounds of message passing, which sees the connections between symbols, terms, literals, and clauses. The GNN is trained on many previous proof searches, estimating which clauses will work together best.

Leapfrogging Our first method implements the idea that the graph-based evaluation of a particular clause may significantly change as new clauses are produced and the context changes. It corresponds to the human-based mathematical exploration, in which initial actions can be done with relatively low confidence and following only uncertain hunches. After some amount of initial exploration is done, clearer patterns often appear, allowing re-evaluation of the approach, focusing on the most promising directions, and discarding of less useful ideas. In tableau-based provers such as leanCoP [15] with a compact notion of a *state*, such methods can be approximated in a reinforcement learning setting by the notion of *big steps* [12] in the Monte-Carlo tree search (MCTS), implementing the standard *explore/exploit* paradigm [7]. In the saturation setting, our proposed algorithm uses short standard saturation runs in the (low-level) exploration phase, after which the set of processed (selected) clauses is re-evaluated and a (high-level, strategic) decision on its most useful subset is made by the GNN. These two phases are iterated in a procedure that we call *leapfrogging*.

In more detail, given a problem consisting of a set of initial clauses $S = S_0$, a saturation-style search (E/ENIGMA) is run on S with an abstract time limit. We may use a fixed limit (e.g., 1000 nontrivial processed clauses) for all runs, or change (e.g. increase) the limits gradually. If the initial run results in a proof or saturation within the limit, the algorithm is finished. If not, we inspect the set of created clauses. We can inspect all generated clauses, or a smaller set, such as the set of all processed clauses. So far, we used the latter because it is typically much smaller and better suits our training methods. This (*large*) set is denoted as L_0. Then we apply a trained graph-based predictor to L_0, which selects a smaller *most promising* subset of L_0, denoted as S_1. We may or may not automatically include also the initial negated conjecture clauses or the whole initial set S_0 in S_1. S_1 is then used as an input to the next limited saturation run of E/ENIGMA. This process is iterated, producing gradually sets S_i and L_i.

[1] Examples of such fair AI-style settings are the global resource limits used in the MPTP Challenge [13] and CASC LTB [18]. Similarly for large ITP benchmarks, e.g., Mizar [11], Flyspeck [10], HOL4 [3], and Isabelle [2].

Table 1: Four leapfrogging runs with different GNN-ENIGMAs

GNN-strategy	original-60s-run	leapfrogging (300-500-60s)	union	added-by-lfrg
G_1	2711	2218	3370	659
G_2	2516	2426	3393	877
G_3	2655	2463	3512	857
G_4	2477	2268	3276	799

Learning Reasoning Components Mathematical problems often have well-separated reasoning and computational components. Examples include numerical calculations, computing derivatives and integrals, performing Boolean algebra in various settings, sequences of standard rewriting and normalization operations in various algebraic theories, etc. Such components of the larger problem can be often solved mostly in isolation from the other components, and only their results are then used together to connect them and solve the larger problem. Human-designed problem solving architectures addressing such decomposition include, e.g., SMT systems, systems such as MetiTarski [1], and a tactic-based learning-guided proof search in systems such as TacticToe [6]. In all these systems, the component procedures or tactics are however *human-designed* and (often painstakingly) human-implemented, with a lot of care both for the components and for the algorithms that merge their results. This seems hard to scale to the large number of complex algorithms and heuristics used in research-level mathematics.

We instead want to *learn* such *targeted components*, expressed as sets of clauses that perform targeted reasoning and computation within the saturation framework.[2] We also want to learn the merging of the results of the components automatically. This is ambitious, but there seems to be growing evidence that such targeted components are being learned in many iterations of GNN-guided proving followed by retraining of the GNNs in our recent large iterative evaluation over Mizar.[3] In these experiments we have significantly extended our previously published results [8],[4] eventually automatically proving 73.5% (more than 40k) of the Mizar theorems. In particular, there are many examples on the project's Github page showing that the GNN is learning to solve more and more involved computations in problems involving differentiation, integration, boolean algebra, algebraic rewriting, etc. Our proposed *Split and Merge* algorithm is therefore to (i) use the GNN to learn to identify interacting reasoning components, (ii) use graph-based and clustering-based algorithms to split the set of clauses into components based on the GNN predictions, (iii) run saturation on the components independently, (iv) possibly merge the most important parts of the components, and (v) iterate. In more detail, we use a modified version of our GNN to predict the graph of future clause interactions in (i), experiment with several (soft) clustering methods for (ii), and again use the GNN to implement (iv).

Experiments The first leapfrogging experiment uses increasing limits on the set of processed clauses (300 and 500) with the final run limited by CPU time (60s). This is done over 28k hard Mizar problems with four differently parameterized GNNs ($G_1, \ldots, G_4$). The methods indeed achieve high complementarity to the original GNN strategies (Table 1), likely thanks to the different context in which the GNNs see the clauses in the subsequent runs. In the first Split and Merge experiment, we get 111 newly solved problems in the Split (component) phase out of 3000 hard problems unsolved in the initial standard saturation run (both using a limit of 1000 processed clauses). Then the Merge phase yields (with the same limit) additional 66 problems.[5] Many of the new proofs indeed show frequent computational patterns (see Appendix A).

[2] Note that this is also *fuzzier* (and possibly easier) than related splitting by *crisp* (neural) conjecturing [5, 19].
[3] https://github.com/ai4reason/ATP_Proofs
[4] The publication of this large evaluation is in preparation.
[5] The full experimental details will be given in the talk.

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Appendix A Interesting Frequent Computational Patterns

Here we show three of the computationally looking Mizar proofs found automatically only by the Split and Merge algorithm for theorems `T16_FDIFF_5`,[6] `T48_NEWTON`,[7] and `T10_MATRIX_4`.[8]

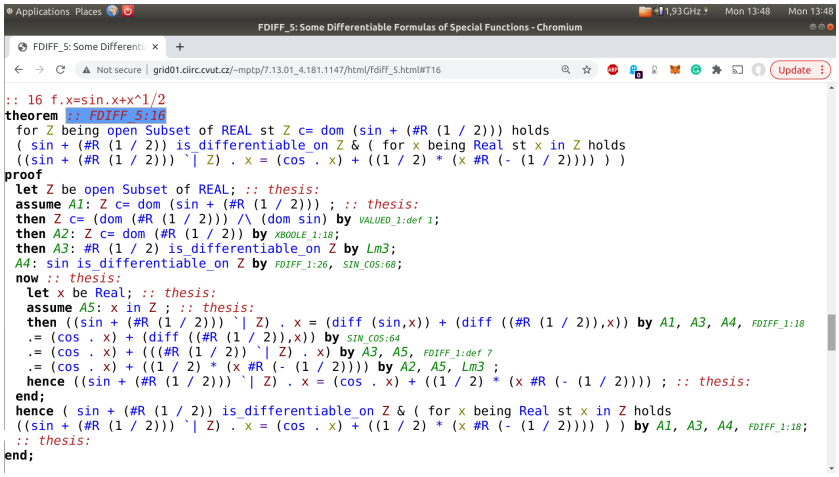

Figure 1: Differentiation – T16_FDIFF_5

[6] http://grid01.ciirc.cvut.cz/~mptp/7.13.01_4.181.1147/html/fdiff_5.html#T16
[7] http://grid01.ciirc.cvut.cz/~mptp/7.13.01_4.181.1147/html/newton.html#T48
[8] http://grid01.ciirc.cvut.cz/~mptp/7.13.01_4.181.1147/html/matrix_4.html#T10

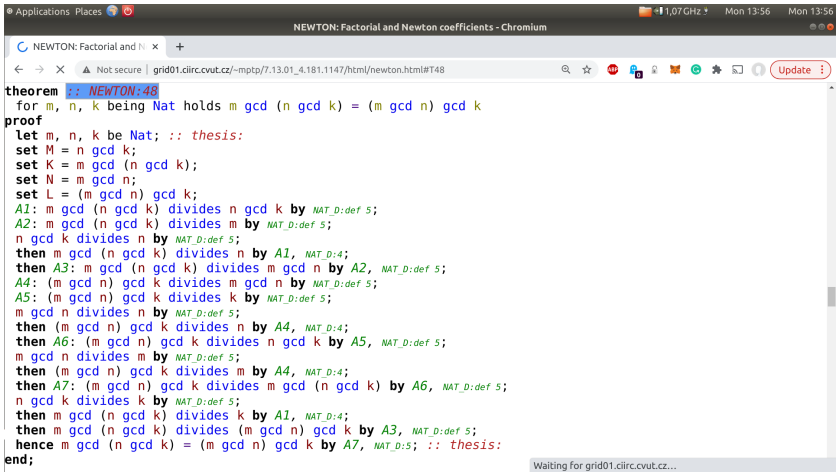

Figure 2: Associativity of gcd by many rewrites – T48_NEWTON

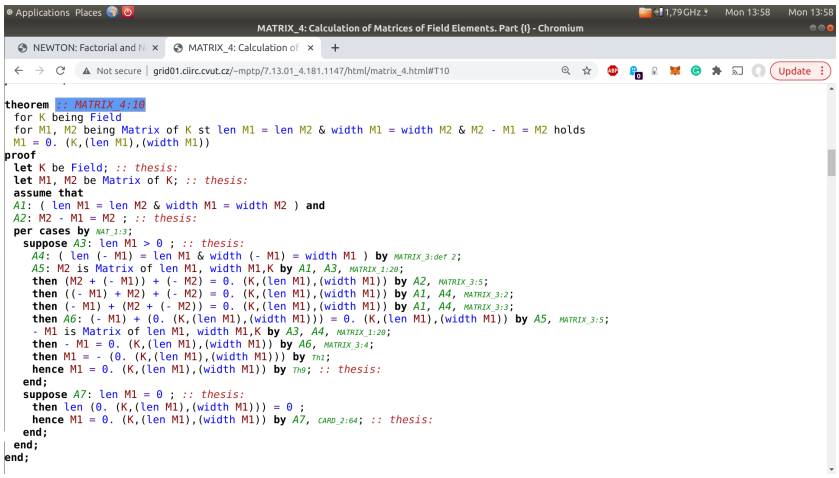

Figure 3: Matrix manipulation – T10_MATRIX_4