Classification of finite semigroups and categories using computational methods *

Wesley Fussner, Najwa Ghannoum, Tomáš Jakl, and Carlos Simpson

Laboratoire J.A. Dieudonné, CNRS, and Université Côte d'Azur
Nice, France
wesley.fussner@unice.fr, najwa.ghannoum@unice.fr, tomas.jakl@unice.fr,
carlos.simpson@unice.fr

Abstract

We report on our work in progress aimed at analyzing the structure of finite categories, with an eye to developing structure theorems for these. In this work, we rely on the use of McCune's Prover9/Mace4 to construct models, providing in particular a count of the number of categories with two non-isomorphic objects and such that all hom-sets have size 3.

The enumeration of associative structures, together with their classification, presents a deep challenge for algebraic combinatorics. The use of computer algebra systems and tools from artificial intelligence have been applied to great success for finite monoids [7], but these techniques have not been applied in the setting of finite categories. Each of the latter consists of a finite set of objects and a finite set of morphisms between them. As associative algebraic structures, these present an avenue for deepening our understanding of enumeration and classification problems in associative algebra. Additionally, a finite category leads to a derived category of modules, and hence to a moduli stack in algebraic geometry [9]. From the classification set, we obtain a range of examples of derived categories and geometric stacks. Thus, another long-term motivation for this work is to try to find examples having interesting geometrical properties by developing a full picture of the combinatorial structure of the classification question.

Our method of analyzing finite categories is to use computational resources to generate statistics regarding finite categories of a certain type, and then analyze this data in terms of certain features of the objects. This yields, *inter alia*, an exact count of the number of categories with two non-isomorphic objects and such that all hom-sets have size 3. The objects of such finite categories may be identified with their endomorphism monoids, and the structure of the categories may be analyzed by reference to properties of these endomorphism monoids.

This method relies on the use of McCune's Prover9/Mace4 [5] to construct models. We offer a representation of finite categories as semigroups equipped with zero and an additional unary predicate, which encodes the identity morphisms. This representation is a generalisation of the well-known Ehresmann-Schein-Nambooripad Theorem which expresses a fundamental connection between inverse semigroups and inductive groupoids (see e.g. [8]). Our representation allows us to restate the enumeration problem for finite categories in terms of these expanded semigroups. Using a program written in Python to generate optimized semigroup equations as an input for Mace4[1], we generate non-isomorphic algebraic models satisfying those equations. This provides a count of the corresponding finite categories.

This work follows the line of research began in [3, 6] and continued in [4], viewing categories as associated to certain square matrices. The entries of the matrix correspond to the size of

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[1] Mace4 is a program that searches for finite models of first-order theories.

hom-sets between every two objects, i.e. the matrix (a_{ij}) corresponds to categories where there are exactly a_{ij} morphisms from i to j. For example, the matrix

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

corresponds to categories with two objects and exactly 3 morphisms between any pair of objects. Viewed from this perspective, our motivation is to study the structure of finite categories for a fixed matrix type. The coefficients on the diagonals constrain the structure and nature of the objects by giving restrictions on their endomorphism monoids, which, when considered as fixed parameters, give insight into the enumeration and classification problem. In particular, we obtain information about the number of categories that can be constructed when the endomorphism monoids of the objects are fixed. Our data shows that some types of endomorphism monoids, such as semilattices, give more options to build categories. This way we also discover which structural properties of finite monoids allow their combinations when realized as objects in categories of a given type.

Allouch and Simpson inaugurated the counting problem in [4], where they count the number of categories associated to matrices whose coefficients are all 2. The calculations in this work are performed by hand, up to matrices of size 3. Using our new methods, we extend the Allouch-Simpson count to the matrices of size 2 whose coefficients are all 3.

Ongoing work by Alfaya, Balzin, and Simpson seeks to apply techniques from neural networks to obtain approximate counts of the number of semigroups of a given cardinality. Because the most difficult combinatorial questions in the present work appear to be related to the constituent semigroups, we expect that similar techniques may further the program of enumeration and classification articulated here.

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