Automated Reasoning for the Andrews-Curtis Conjecture

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Abstract

We present recent developments in the applications of automated theorem proving and disproving in the investigation of the Andrews-Curtis conjecture. We answer negatively open question from [9] on extensions of AC-transformations and demonstrate trivializations of the cases reported in *op.cit.* We outline further directions of the research on the borders between Mathematics, Automated Reasoning and, more generally, AI.

Introduction and Outline

The Andrews-Curtis conjecture is a well-known open conjecture [1] in low-dimensional topology and combinatorial group theory. In terms of the latter it states that every balanced presentation of the trivial group can be transformed into a trivial one by a series of simple transformations including Nielsen transformations and conjugations of relators. Many authors expressed belief that ACC is likely to be false and there are series of trivial group presentations for which conjectured trivializations remain unknown. Various computational techniques have been applied for the search of the required trivializations (i.e. for elimination of potential counterexamples for ACC), including the methods from *computational group theory* [2]; *genetic algorithms* [8, 10, 5]; *systematic breadth-first search algorithms* [3] to name a few.

In [6, 7] we have proposed to use automated deduction in first-order logic in the search of trivializations and have shown that the approach is very competitive. It was able to trivialize any known to us example tackled by any alternative method reported in the literature, and in [7] we demonstrated new examples of simplifications, including for group presentations of dimension[1] 3 and 4. In our approach we formalized the AC transformations (commonly presented at the meta-level) at the object level of term rewriting (modulo group theory) and first-order deduction. The problem of finding AC-simplifications is reduced here to the problem of proving first-order formulae, which is then delegated to the available automated theorem provers. We also have shown that the disproving by automated finite countermodel finding can be used to show the impossibility of trivializations by subsystems of AC transformations.

In recent work [9][2] the authors proposed a novel algorithmic approach to AC simplifications which relies on the use of generalized moves and a strong equivalence relation on group presentations. Further they have shown that for the famous open series of Akbulut-Kirby presentations $AK(n), n \geq 3$ extending the AC rules by automorphisms of free group F_2 does not increase the sets of reachable presentations. They notice that in general "It is not known if adding these transformations to AC-moves results in an equivalent system of transformations or not".

In this paper:

- We answer the open question from [9] negatively, that is adding automorphic transformations to AC rules leads indeed to non-equivalent systems of transformations. We utilize the approach from [7] and use automated disproving by finite countermodel finding

[1] that is the number of generators (= number of relators for balanced presentations)
[2] which we neglected to refer to and compare with in [7]

- We show that all 12 novel AC trivialization cases reported in [9], Table 4 can also be tackled by our automated theorem proving method,
- We report that we failed to find[3] AC-transformations from trivial group presentations to their images under F_2 automorphisms, reported in [9], Table 3, by our automated theorem proving method. That means that unlike for trivializations finding general AC-transformations by generic theorem proving may not be that efficient as by specialized methods of [9].

Discussion

We have shown that generic automated first-order proving and disproving can be used in combinatorial group theory, both in tackling open questions and as a competitive alternative to specialized algorithms. That places the reported research at the borders between Mathematics, Automated Reasoning and more generally, Artificial Intelligence. Further directions include development of automated extraction of the simplifying move sequences from proofs; machine learning applied to AC-proofs, in a spirit of, for example [4]; investigation of how first-order proving and disproving methods and strategies affect the efficiency of automated reasoning in this area.

We have published the extended version of this paper and all computer-generated proofs and countermodels online[4].

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