Learning to Prove with Tactics

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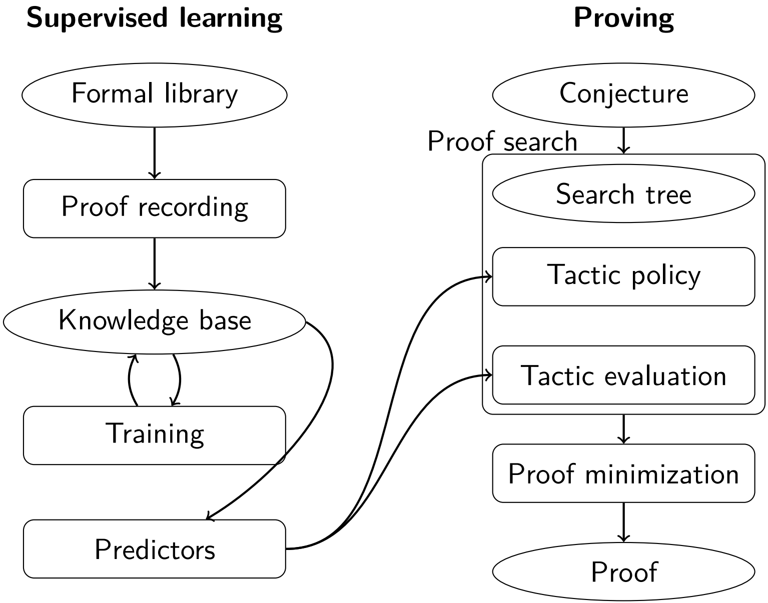
Problem

Can we formally prove mathematical formulas automatically?

Our solution



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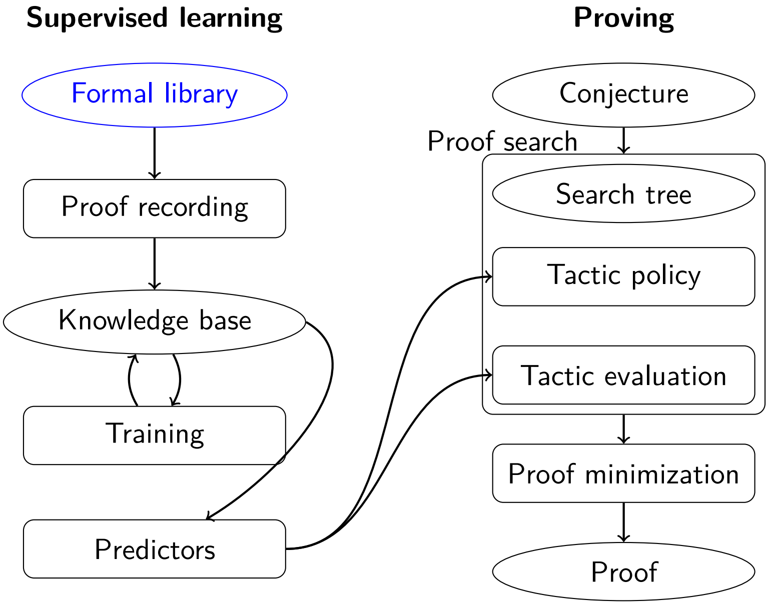
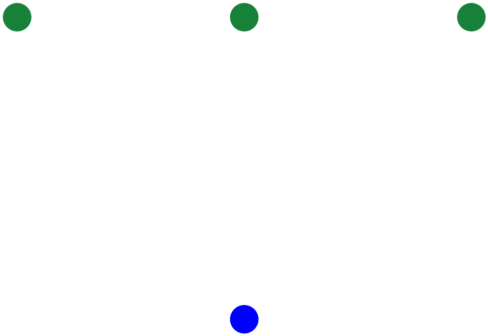
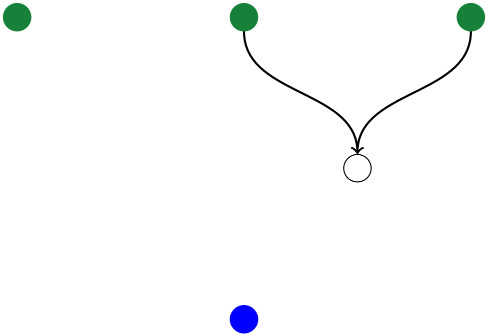


Formal library: reasoning with inference rules



● axiom
● conjecture
→ rule
○ lemma

Formal library: reasoning with inference rules



- axiom
- conjecture
- $\rightarrow$ rule
- lemma

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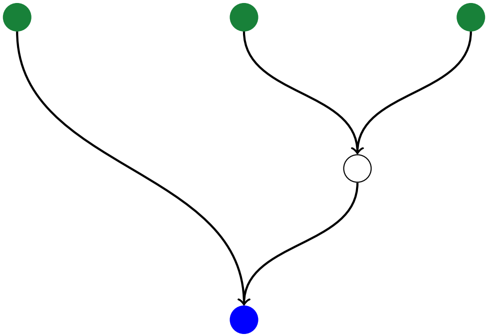

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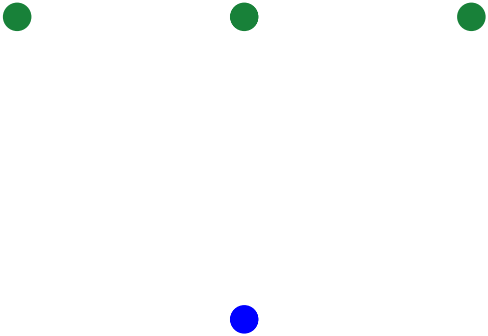
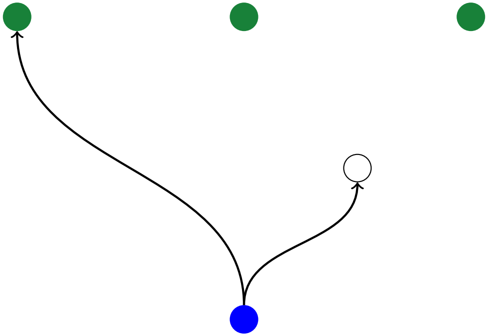
Formal library: reasoning with tactics



- axiom
- conjecture
- → tactic
- goal

Formal library: reasoning with tactics



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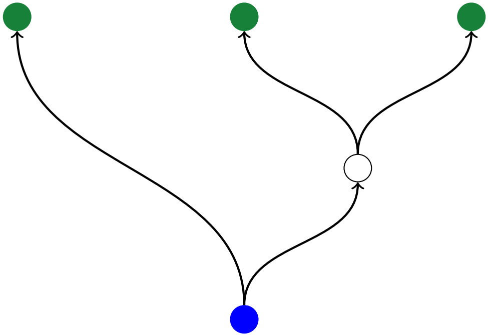
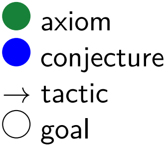
Formal library: reasoning with tactics





Formal library: tactics

```
REWRITE_TAC

INDUCT_TAC

METIS_TAC
```

Formal library: composing tactics

`THENL` tactical composes the effect of tactics.

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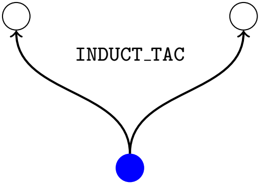

Formal library: composing tactics

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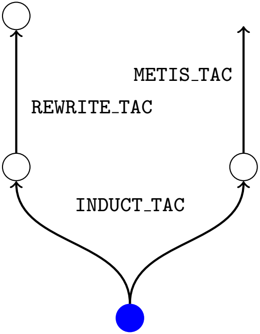

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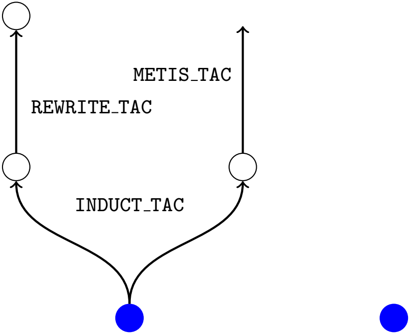
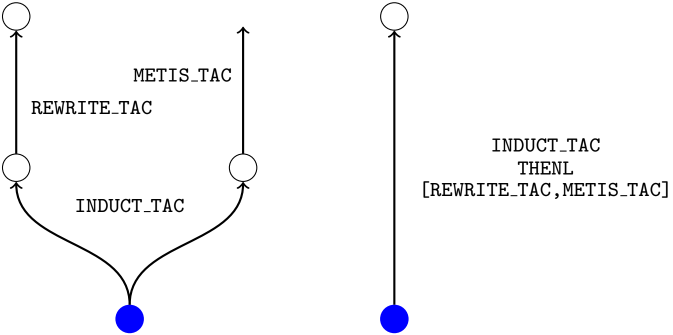


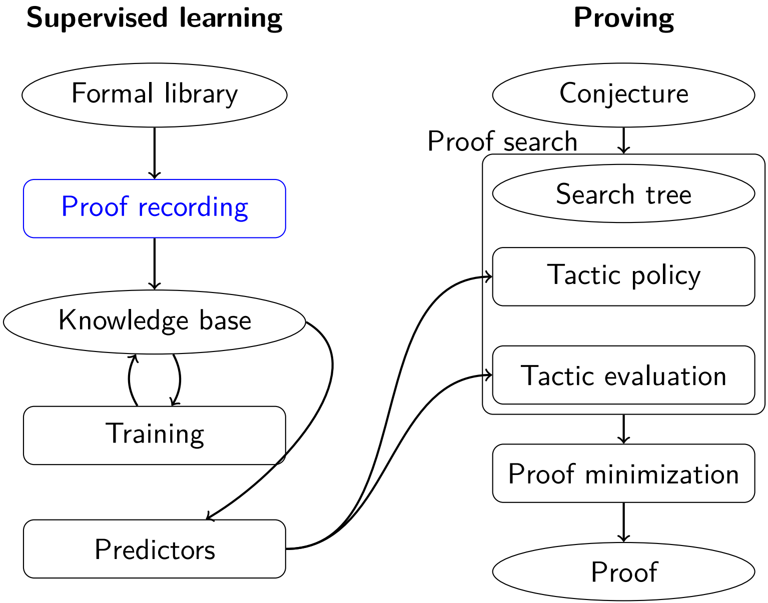
Formal library: composing tactics

THENL tactical composes the effect of tactics.



Demo

Plan



Supervised learning
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Proof recording
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Training
Predictors
Conjecture
Proof search
Search tree
Tactic policy
Tactic evaluation
Proof minimization
Proof

Recording

Original proof:

```
INDUCT_TAC THENL [REWRITE_TAC, METIS_TAC]
```

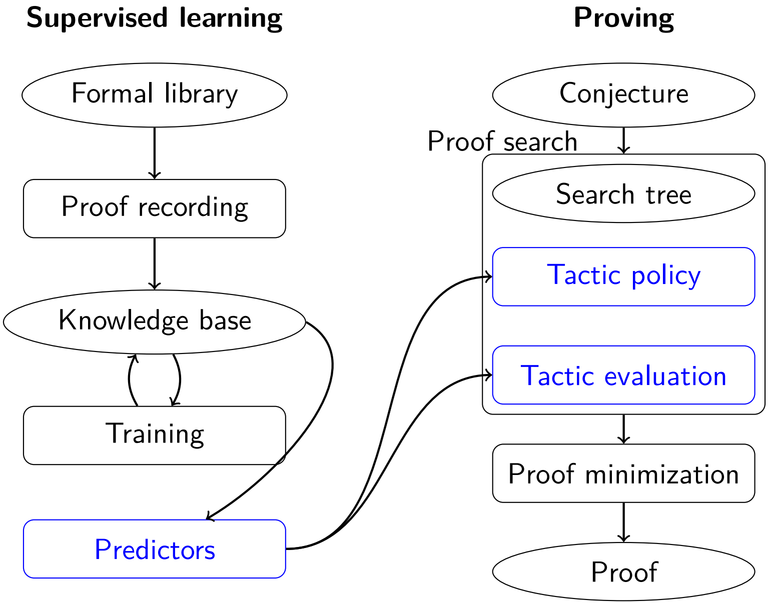
Modified proof:

```
(R numLib.INDUCT_TAC) THENL
  [R boolLib.REWRITE_TAC, R metisLib.METIS_TAC]
```

Database of tactics:

```
R (f n) (f (SUC n)) ⇒ transitive R: INDUCT_TAC
n ∗ m ≤ n ∗ p ⇒ (n = 0) ∨ m ≤ p    : REWRITE_TAC
INJ f U(:num) s ⇒ INFINITE s       : METIS_TAC
...
```

Plan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Prediction algorithm

Algorithm:
Nearest neighbor weighted by TF-IDF heuristics

Effect:
Order goals from the database according to their distance to a target goal.

Remark:
This is algorithm performs premise selection.
How do we adapt it to predict tactics?

Policy: choosing a tactic

Database of tactics is a map from goals to tactics.

```
R (f n) (f (SUC n)) ⇒ transitive R: INDUCT_TAC
n * m ≤ n * p ⇒ (n = 0) ∨ m ≤ p    : REWRITE_TAC
INJ f U(:num) s ⇒ INFINITE s       : METIS_TAC
```

...

An order on goals induces an order on tactics.

New goal appearing during proof search:

```
LENGTH (MAP f l) = LENGTH l
```

Policy for the new goal:

Rank	Tactic	Policy
1	REWRITE_TAC	0.5
2	METIS_TAC	0.25
...		
4	INDUCT_TAC	0.0625
...		

Value function: provability of a list of goals

Database of lists of goals:

- Positive examples: appears in human proofs.
- Negative examples: produced during TacticToe search but do not appear in the final proof.

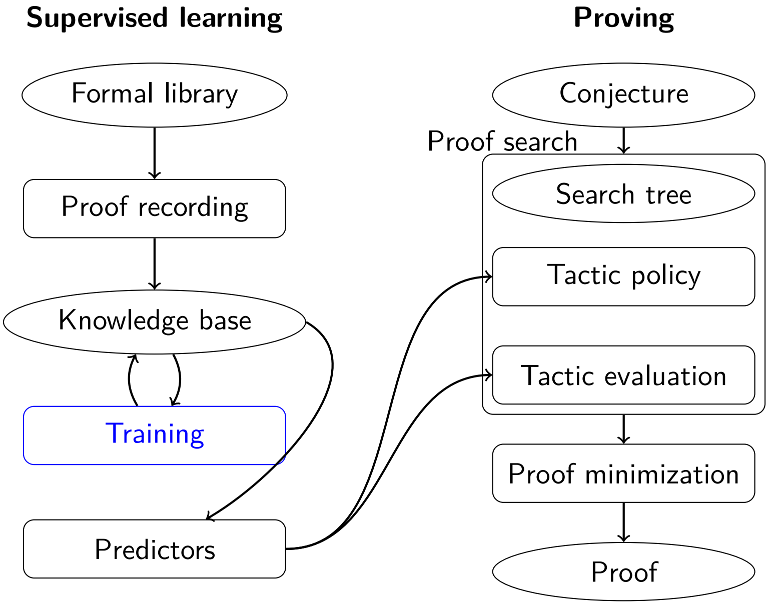
Value function:

Percentage of positives in the 10 closest lists of goals of a target list of goals.

Future work:

Estimate the number of steps needed to prove a list of goals.

Plan



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Training

Improve recorded data to create better predictions during search.

Training: orthogonalization

Issue: Many tactics are doing the same job on a goal g.

Solution: Competition for g where the most popular tactic wins.

Training: orthogonalization

Recorded goal-tactic pair:

```
LENGTH (MAP f l) = LENGTH l: INDUCT_TAC
```

Competition:

	Progress	Coverage
INDUCT_TAC	Yes	136
REWRITE_TAC	No	2567
METIS_TAC	Yes	694

Added to the database:

```
LENGTH (MAP f l) = LENGTH l: METIS_TAC
```

Result: 6 % improvement.

Training: abstraction

Issue: Some theorems are never used inside tactics.

Solution: Abstract all lists of theorems in a tactic
and instantiate them depending on the target goal.

Training: abstraction

Abstraction algorithm:

```
Original      : REWRITE_TAC [T1,T2]
Abstraction   : REWRITE_TAC X
Instantiation: REWRITE_TAC [T67, T1, T43, ..]
```

Question: Dow we keep the original or the abstraction ?

Answer: Let them compete during orthogonalization.

Training: preselection

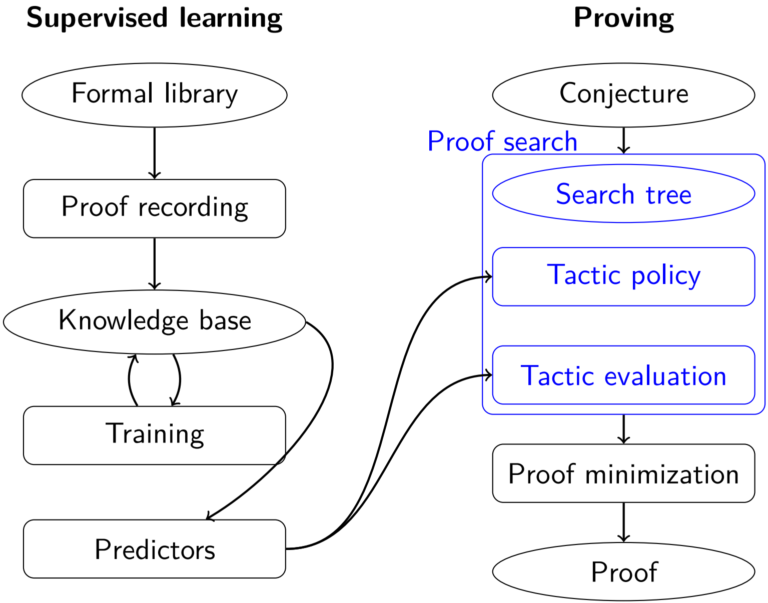
Issue: Predictions are too slow during proof search.

Solution: Preselect 1000 suitable tactics using dependencies.

Dependency: Appear in the same proof.

Result: 15% improvement.

Plan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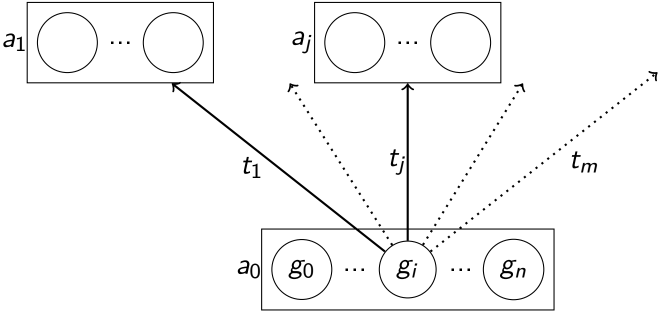
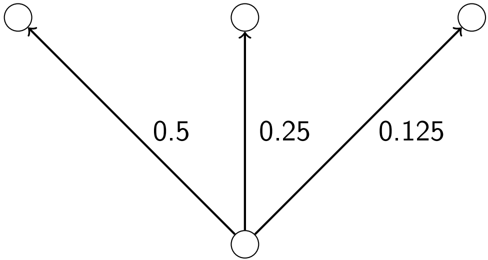
Proof search: search tree



Proof search: advanced tree search



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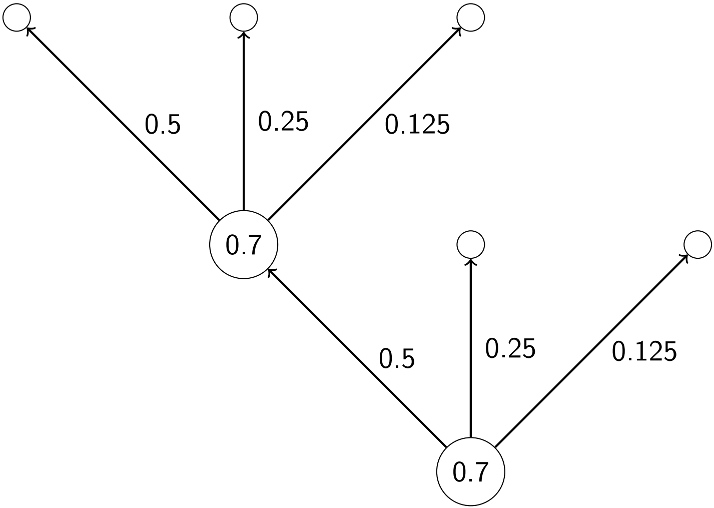

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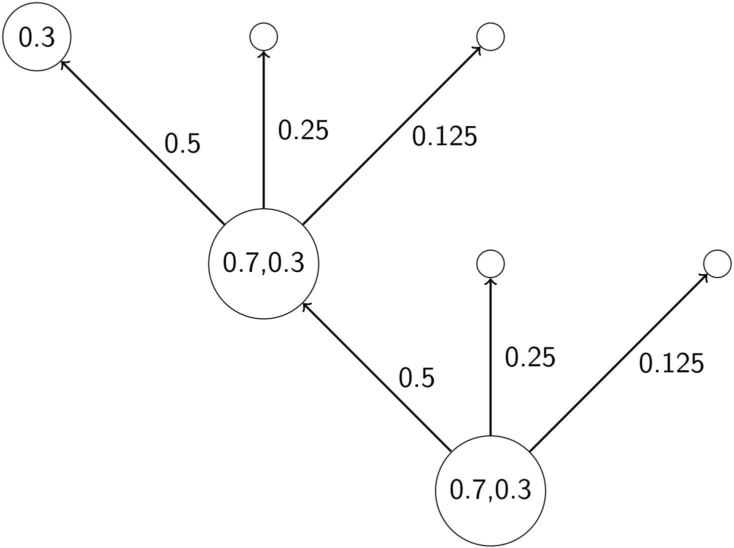

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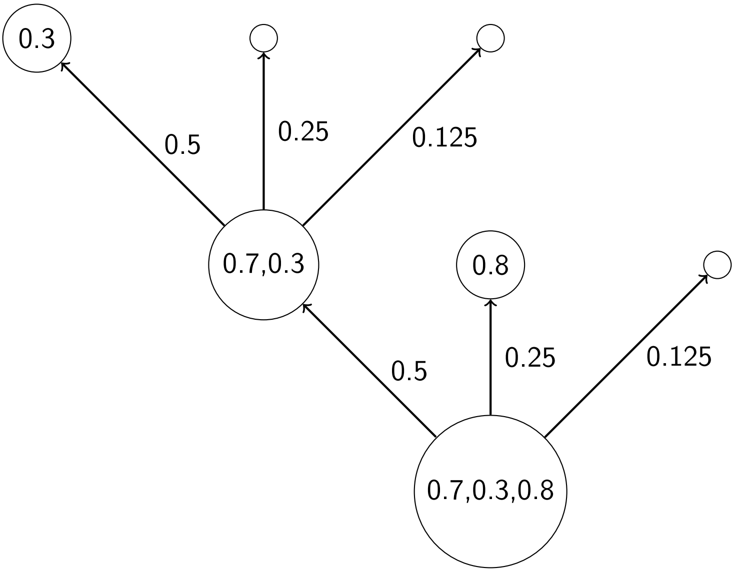


Re-proving: results

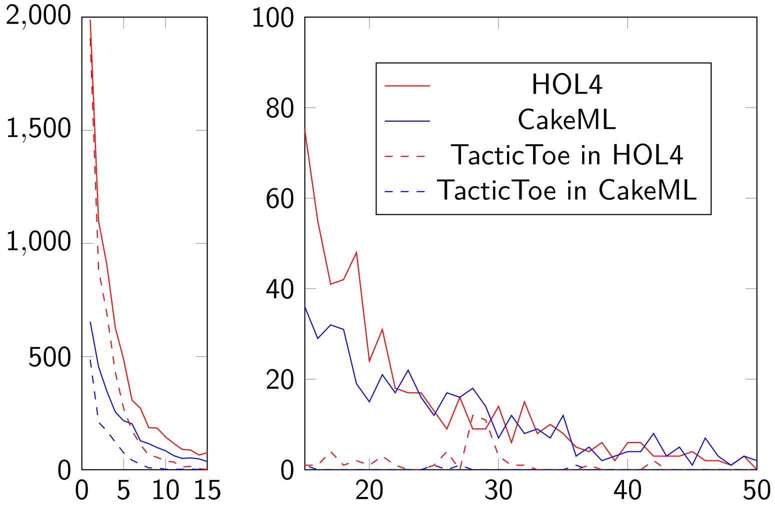
Evaluation is "fair". (not totally if you ask Freek)

Only previous proofs are available for training.

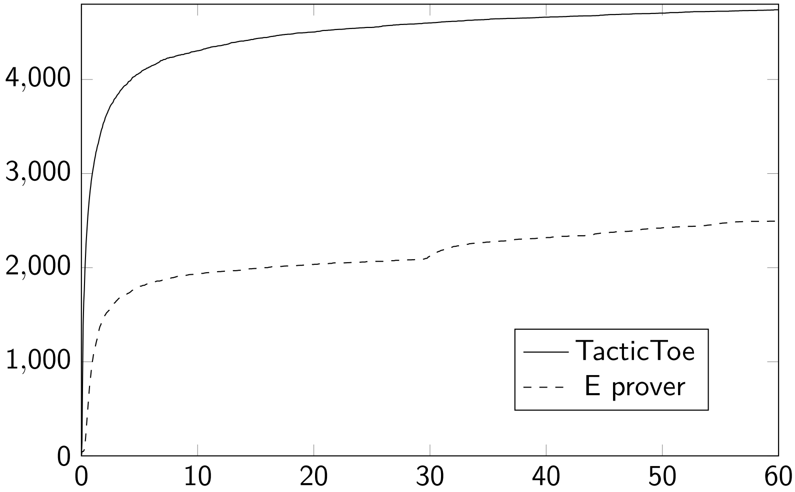
TacticToe does not call external provers.

	HOL4: 7164, 60s	CakeML: 3329, 15s
E prover	2472 (34.5%)	
TacticToe	4760 (66.4%)	1161 (34.9%)
Total	4946 (69.0%)	

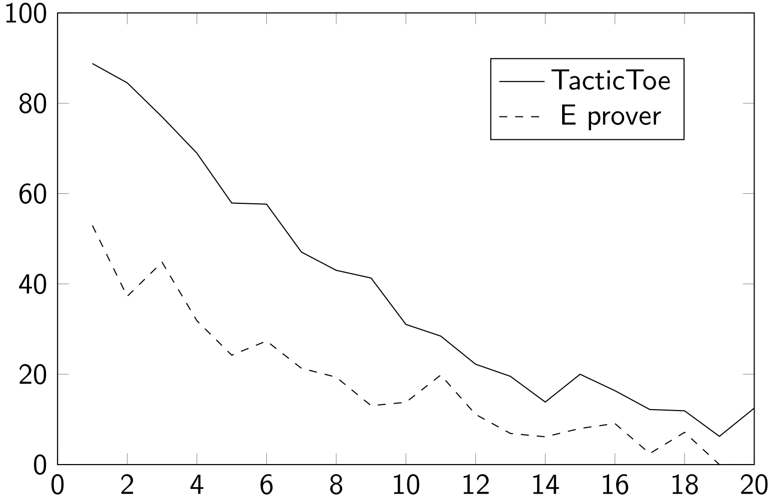
Re-proving: proofs of size x



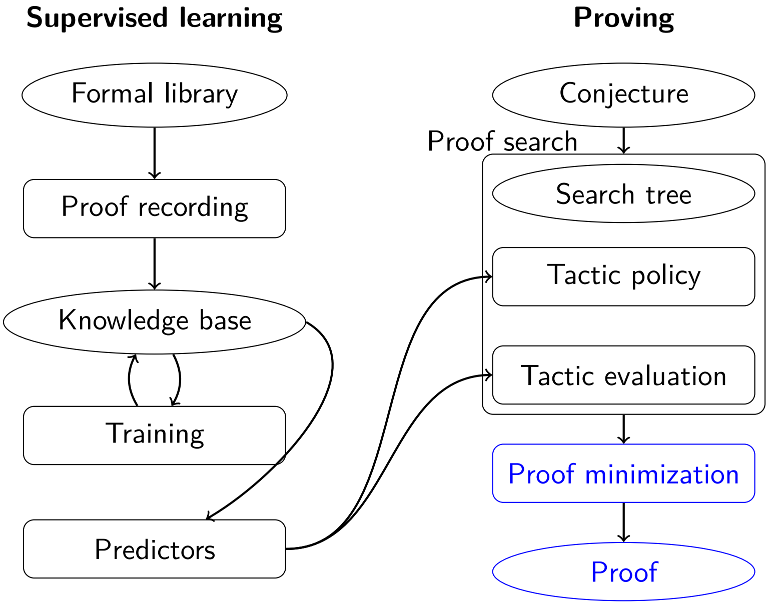
Re-proving: HOL4 proofs found in less than x seconds



Re-proving: percentage of solved HOL4 proof of size x



Plan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Minimization and embellishment

Raw proof:

```
boolLib.REWRITE_TAC [DB.fetch "list" "EVERY_CONJ",... ]
  THEN
BasicProvers.Induct_on [HolKernel.QUOTE "l"]
  THENL
  [BasicProvers.SRW_TAC [] [],
   simpLib.ASM_SIMP_TAC (BasicProvers.srw_ss ())
   [boolLib.DISJ_IMP_THM, DB.fetch "list" "MAP",
    DB.fetch "list" "CONS_11", boolLib.FORALL_AND_THM]]
```

Processed proof:

```
Induct_on `l` THENL
  [SRW_TAC [] [],
   ASM_SIMP_TAC (srw_ss ())
   [DISJ_IMP_THM, FORALL_AND_THM]]
```

Conclusion

Summary: TacticToe learns from human proofs to solve new goals.

Advantages over ATPs (E prover) for ITP (HOL4) users:

- Includes domain specific automation found in the ITP.
- Generated proofs are human-level proofs.
- No translation or reconstruction needed.

Future work

Enlarge the action space:
parameter synthesis, sequence of tactics, forward proofs.

Train tactics by evaluating input/output pairs.

Conjecture intermediate lemmas.