Towards Machine Learning for Quantification

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Outline

Intro: QBF, Expansion, Games, Careful expansion

Solving QBF

Learning in QBF

Bernays–Schönfinkel (“Effectively Propositional Logic”) — Finite Models

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- Alternatively, use equality:

$$t \neq f \wedge (\forall X_u.\ (X_u = t) \leftrightarrow p_e(X_u))$$

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  Example

$$\forall u \exists e. (u \leftrightarrow e)$$

$\exists$-player wins by playing $e \triangleq u$.

Solving QBF

$$\exists \mathcal{E}\, \forall \mathcal{U}.\ \phi \equiv \exists \mathcal{E}.\ \bigwedge_{\mu \in 2^{\mathcal{U}}} \phi[\mu]$$

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Can be solved by SAT $\left(\bigwedge_{\mu \in 2^{\mathcal{U}}} \phi[\mu]\right)$. Impractical!

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Observe:

$$\exists \mathcal{E}.\, \bigwedge_{\mu \in 2^{\mathcal{U}}} \phi[\mu] \Rightarrow \exists \mathcal{E}.\, \bigwedge_{\mu \in \omega} \phi[\mu]$$

for some $\omega \subseteq 2^{\mathcal{U}}$

What is a good ω?

Solving by CEGAR Expansion Contd.

$$\exists \mathcal{E}\, \forall \mathcal{U}.\ \phi \equiv \exists \mathcal{E}.\ \bigwedge_{\mu \in 2^{\mathcal{U}}} \phi[\mu]$$

Expand gradually instead: [Janota and Marques-Silva, 2011]

- Pick τ_0 arbitrary assignment to $\mathcal{E}$

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- $\mathsf{SAT}(\neg\phi[\tau_1]) = \mu_2$ assignment to $\mathcal{U}$
- $\mathsf{SAT}(\phi[\mu_0] \wedge \phi[\mu_1]) = \tau_2$ assignment to $\mathcal{E}$

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- $\mathsf{SAT}(\neg\phi[\tau_1]) = \mu_2$ assignment to $\mathcal{U}$
- $\mathsf{SAT}(\phi[\mu_0] \wedge \phi[\mu_1]) = \tau_2$ assignment to $\mathcal{E}$
- After n iterations

$$\exists \mathcal{E}.\ \bigwedge_{i \in 1..n} \phi[\tau_i]$$

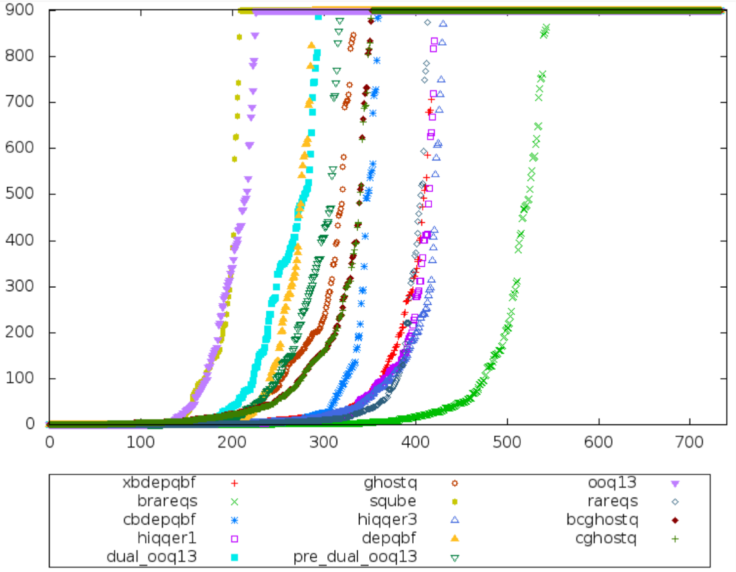
Abstraction-Based Algorithm for a Winning Move

Algorithm for $\exists\forall$. Generalize to arbitrary number of alternations using recursion. [Janota et al., 2012].

Function `Solve`($\exists X \forall Y.\, \phi$)

$\alpha \leftarrow$ true // start with an empty abstraction
while true **do**
  $\tau \leftarrow \mathsf{SAT}(\alpha)$ // find a candidate
  if $\tau = \bot$ **then return** $\bot$
  $\mu \leftarrow$ `Solve`($\neg\phi[X \leftarrow \tau]$) // find a countermove
  if $\mu = \bot$ **then return** τ
  $\alpha \leftarrow \alpha \wedge \phi[Y \leftarrow \mu]$ // refine abstraction

Results, QBF-Gallery ’14, Application Track



Careful Expansion: Good Example

$$\exists x \ldots \forall y \ldots . \phi \wedge y$$

Setting countermove $y \leftarrow 0$ yields false. Stop.

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$\exists x \ldots \forall y \ldots .\ x \vee \phi$

Setting candidate $x \leftarrow 1$ yields true (impossible to falsify). Stop.

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5. $\mathsf{SAT}(x \Leftrightarrow 0 \wedge x \Leftrightarrow 1) \ldots$ UNSAT Stop

Careful Expansion: Ugly Example

$$\exists x_1 x_2 \forall y_1 y_2.\ x_1 \Leftrightarrow y_1 \vee x_2 \Leftrightarrow y_2$$

1. $x_1, x_2 \leftarrow 0, 0$

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6. $\ldots$

Learning in QBF

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- Idea: instead of plugging in constants, plug in functions.
- **Where do we get the functions?**

Use Machine Learning

[Janota, 2018]

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4. Repeat.
5. Additional heuristic: If a learned function still works, keep it. “Don’t fix what ain’t broke.”

x_1	x_2	...	x_n	y_1	y_2	...	y_n
0	0	...	0	1	1	...	1
1	0	...	0	0	1	...	1
0	0	...	1	1	1	...	0
0	1	...	1	1	0	...	0

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- After 2 steps: $y_1 \leftarrow \neg x_1$, $y_i \leftarrow 1$ for $i \in 2..n$.

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- After 4 steps: $y_1 \leftarrow \neg x_1$ $y_2 \leftarrow \neg x_2 \ldots$
- Eventually we learn the right functions.

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- Recursion to generalize to multiple levels as before.
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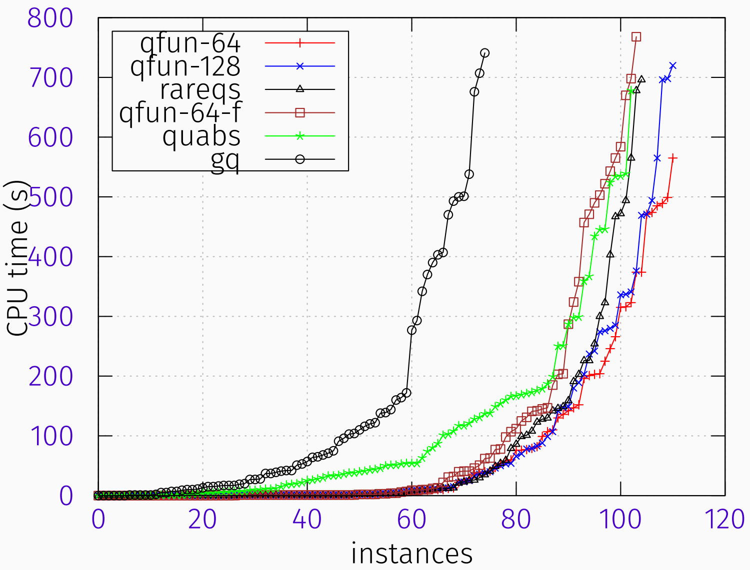
- Learning using decision trees by ID3 algorithm.

Current Implementation: Experiments



Bernays–Schönfinkel (“Effectively Propositional Logic”) — Finite Models

Bernays–Schönfinkel (EPR)

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- ϕ uses predicates $p_1, \ldots, p_m$ and constants $c_1, \ldots, c_n$.
- Finite model property: formulas has a model iff it has a model of size $\leq n$.
- Therefore we can look for a model with the universe $*_1, \ldots, *_{n'}$, $n' \leq n$.

$$\exists p_1 \ldots p_m \exists c_1 \ldots c_n \forall X.\ \phi$$

p_i predicates, c_i constants, X variables

1. $\alpha \leftarrow \mathtt{true}$

$$\exists p_1 \ldots p_m \exists c_1 \ldots c_n \forall X.\ \phi$$

p_i predicates, c_i constants, X variables

1. $\alpha \leftarrow$ `true`
2. Find interpretation for α: $\mathcal{I} \leftarrow \mathtt{SAT}(\alpha)$

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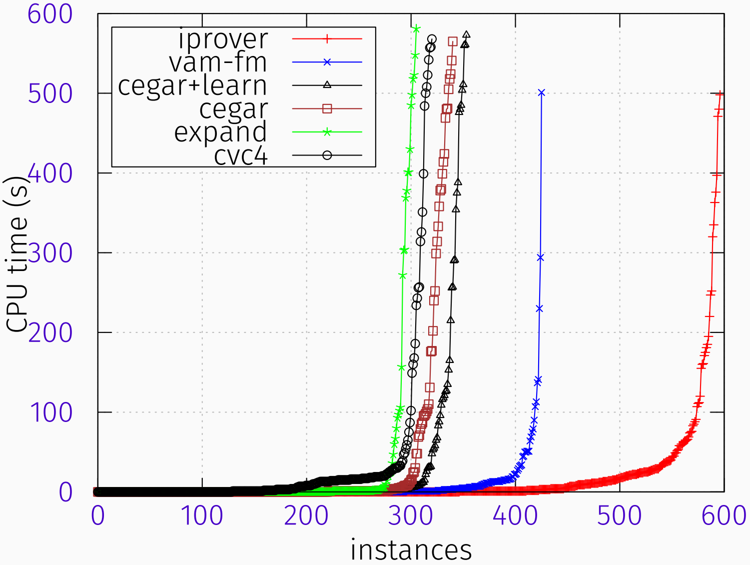
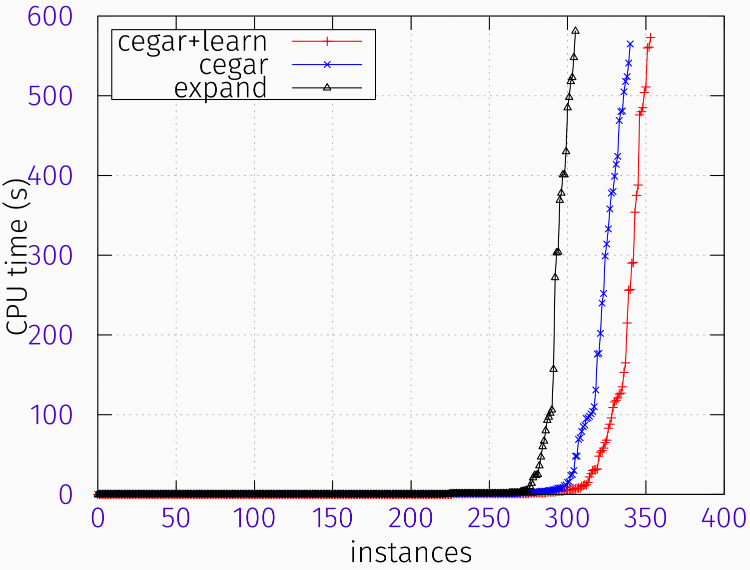
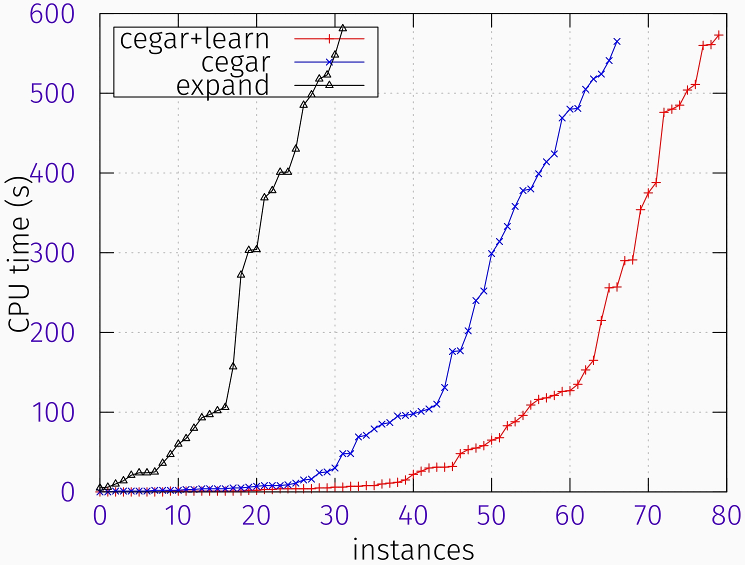
5. Learn: $t \triangleq *_1$, $p(X_1, \ldots, X_n) \triangleq (X_1 = *_1)$,


iprover
vam-fm
cegar+learn
cegar
expand
cvc4
CPU time (s)
0
100
200
300
400
500
600
0
100
200
300
400
500
600
instances

Preliminary Results





cegar+learn
cegar
expand
CPU time (s)
instances
0
100
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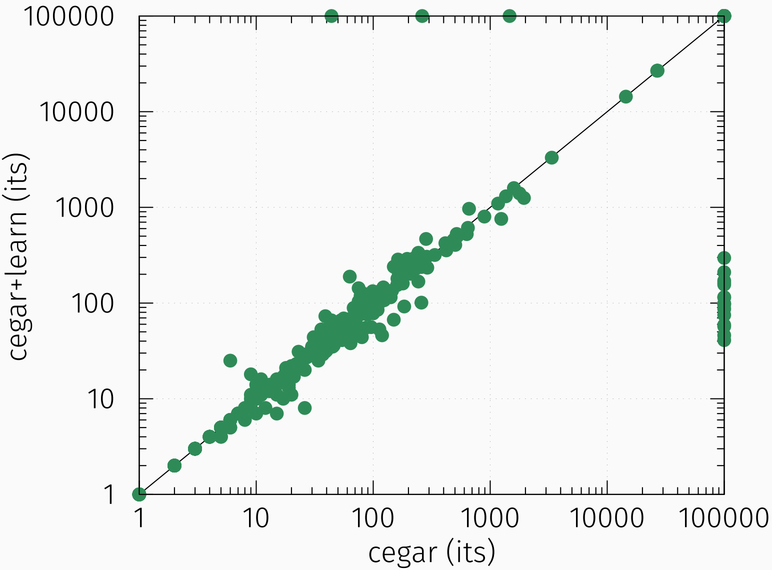
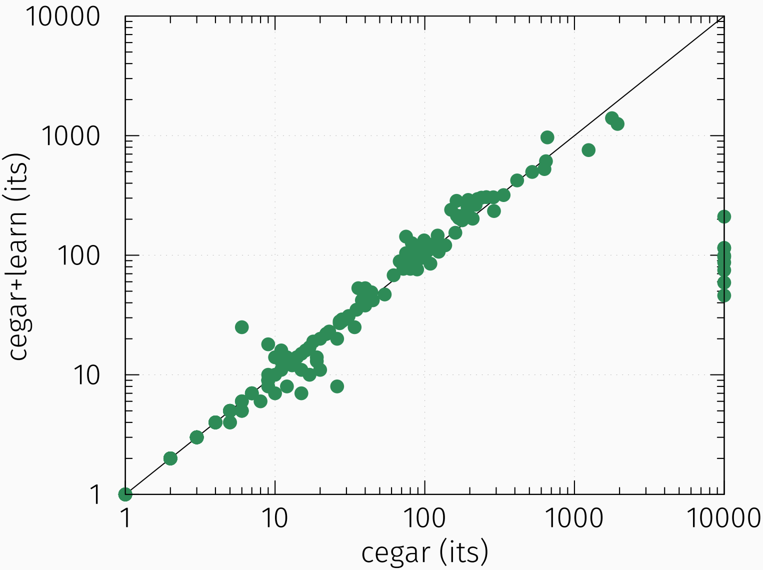
80

Learn vs. CEGAR, Iterations



Learn vs. CEGAR, Iterations — Only True



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Thank You for Your Attention!

Questions?

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