Solving Natural Language Math Problems

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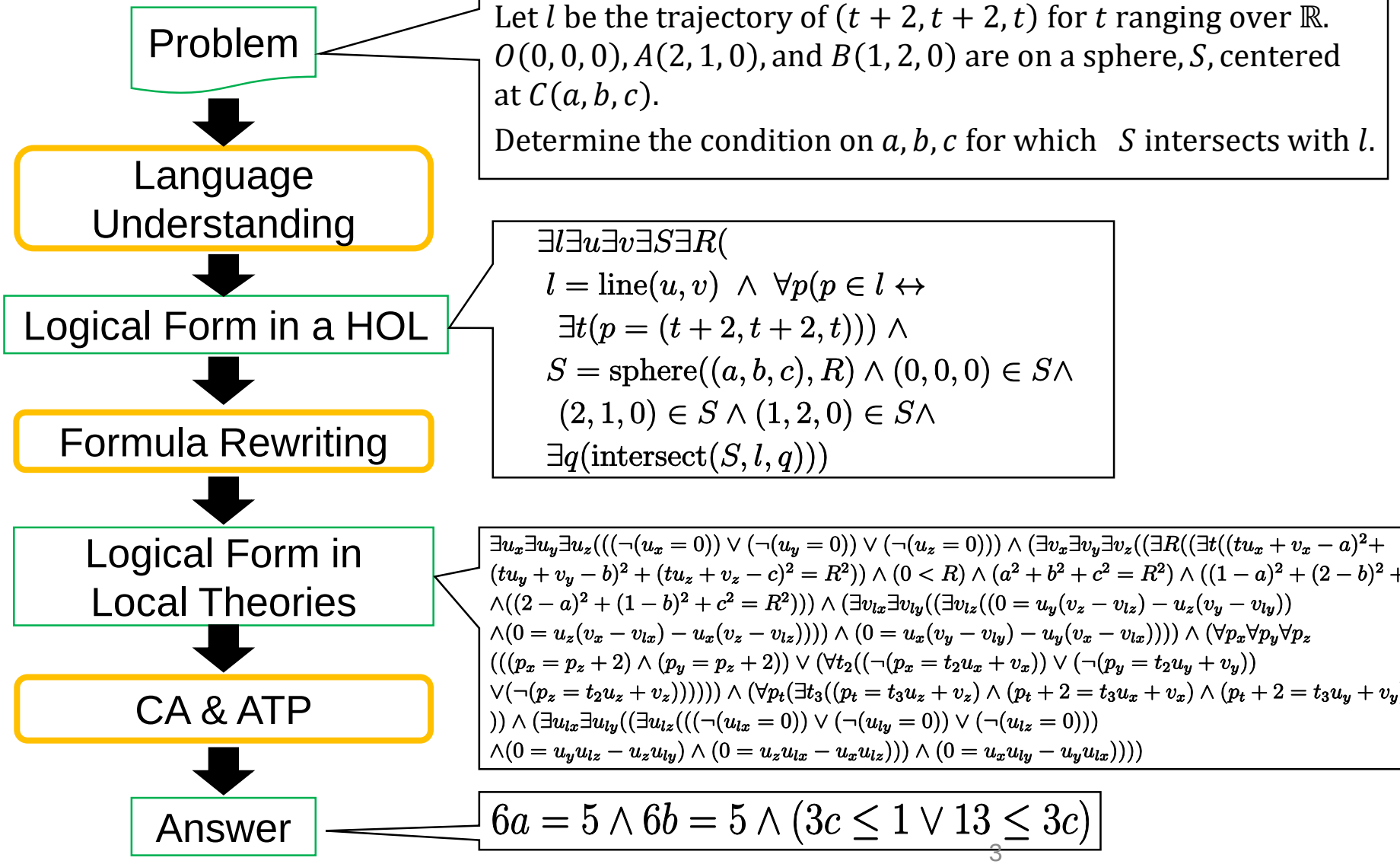
Solving NL Math – why?

- It is the first and the last goal of symbolic approach to language understanding (LU)
  - Formalization of the domain is the prerequisite for LU
- Problem solving is the only way to compare different LU systems
  - Only the input and output are observable
  - No ground-truth for a mid-layer’s output

System Overview



Today's Topics

- Parsing Math Problem Text with Combinatory Categorial Grammar
- Benchmarking a CAS-based solver with formalized pre-university math problems

Combinatory Categorial Grammar

- Word ⇔ (syntactic category, λ-expression)

Word type	Example
Proper noun	"John" ⇔ (NP, john)
Common noun	"cat" ⇔ (N, λx.cat(x))
Intransitive verb	"runs" ⇔ ($S \backslash NP$, λx.run(x))
Transitive verb	"loves" ⇔ ($S \backslash NP/NP$, λy.λx.love(x,y))
Indefinite article	"a" ⇔ ($S/(S \backslash NP)/N$, λN.λP.∃x(Nx∧Px))
Quantifier	"every" ⇔ ($S/(S \backslash NP)/N$, λN.λP.∀x(Nx→Px))

Combinatory rules

Forward application

$$>\ \frac{X/Y : f \quad Y : y}{X : f\ y}$$

Backward application

$$<\ \frac{Y : y \quad X\backslash Y : f}{X : f\ y}$$

Forward composition

$$>B\ \frac{X/Y : f \quad Y/Z : g}{X/Z : \lambda z.f\ (g z)}$$

etc.

$$\dfrac{\dfrac{\text{John}}{NP : \text{john}} \quad \dfrac{\dfrac{\text{loves}}{S\backslash NP/NP : \lambda x.\lambda y.\text{love}(y,x)} \quad \dfrac{\dfrac{\text{a}}{S\backslash NP\backslash(S\backslash NP/NP)/N : \lambda N.\lambda P.\lambda y.\exists x(Nx \wedge Pxy)} \quad \dfrac{\text{cat}}{N : \lambda x.\text{cat}(x)}}{S\backslash NP\backslash(S\backslash NP/NP) : \lambda P.\lambda y.\exists x(\text{cat}(x) \wedge Pxy)}}{S\backslash NP : \lambda y.\exists x(\text{cat}(x) \wedge \text{love}(y,x))}}{S : \exists x(\text{cat}(x) \wedge \text{love}(\text{john},x))}$$

Combinatory rules

Forward application $>\dfrac{X/Y : f \quad Y : y}{X : f\,y}$ Backward application $<\dfrac{Y : y \quad X\backslash Y : f}{X : f\,y}$ Forward composition $>B\dfrac{X/Y : f \quad Y/Z : g}{X/Z : \lambda z.f\,(g\,z)}$ etc.

$$\dfrac{\dfrac{\text{John}}{NP : \text{john}} \quad < \dfrac{\text{loves}}{S\backslash NP/NP : \lambda x.\lambda y.\text{love}(y,x)} \quad > \dfrac{\dfrac{\text{a}}{S\backslash NP\backslash(S\backslash NP/NP)/N : \lambda N.\lambda P.\lambda y.\exists x(Nx \wedge Pxy)} \quad \dfrac{\text{cat}}{N : \lambda x.\text{cat}(x)}}{S\backslash NP\backslash(S\backslash NP/NP) : \lambda P.\lambda y.\exists x(\text{cat}(x) \wedge Pxy)}}{S\backslash NP : \lambda y.\exists x(\text{cat}(x) \wedge \text{love}(y,x))}}{S : \exists x(\text{cat}(x) \wedge \text{love}(\text{john},x))}$$

Syntactic Category = Semantic Type + Syntactic Constraints

Example

"distance" (as in "distance between *P* and *Q*")

- Syntactic cat.: $NP_{\texttt{Real}}/PP_{between,\texttt{(Point,Point)}}$
- Semantic function: `λp.dist(p)`
- Semantic type: `(Point, Point)` → `Real`

$$\dfrac{\dfrac{\text{distance}}{NP_{Real}/PP_{btwn,(Pnt,Pnt)} : \texttt{λp.dist(p)}} \quad \dfrac{\dfrac{\text{between}}{PP_{btwn,(\alpha,\beta)}/NP_{(\alpha,\beta)} : \texttt{id}} \quad \dfrac{\dfrac{P}{NP_{Pnt} : \texttt{P}} \quad \dfrac{\text{and}}{NP_{(\alpha,\beta)} \backslash NP_{\alpha}/NP_{\beta} : \texttt{λy.λx.(x,y)}} \quad \dfrac{Q}{NP_{Pnt} : \texttt{Q}}}{NP_{(Pnt,Pnt)} : \texttt{(P,Q)}}}{PP_{btwn,(Pnt,Pnt)} : \texttt{(P,Q)}}}{NP_{Real} : \texttt{dist(P,Q)}}$$

Comparison with compilers

- Compilers : source code → machine code
- NL parsing : math problem → logical form

- NL parsing = type check
  + syntax check
  + denotational semantics

- Besides, the grammar is only partially known and ambiguous

Grammar and lexicon: current status

- Size
  - 31 combinatory rules
  - 6,652 different word forms
  - 42,154 triples of <word, category, λ-term>

- What's not in textbook (toy) grammars:
  - Imperatives, pluralities, relation/attribute nouns, context dependent semantics, action verbs, etc.

- Coverage:
  - 70%~80% of university math exam sentences can be parsed (either correctly or wrongly)

Remaining issues

- Lexicon / grammar coverage

- Hypothesis explosion due to local ambiguity
  - "$y = ax^2$": equality or $\lambda x.ax^2$ or $\{ (x,y) \mid y = ax^2 \}$
  - "if A then B and C": $(A \rightarrow B) \,\&\, C$ or $A \rightarrow (B \,\&\, C)$

- Inter-sentential logical structure analysis. E.g.,
  - Sentence 1: If A then B.
  - Sentence 2: If C then D.

  - $(A \rightarrow B) \,\&\, (C \rightarrow D)$
  - $A \rightarrow (B \,\&\, (C \rightarrow D))$
  - $(A \rightarrow B) \,\&\, (A \rightarrow (B \,\&\, C) \rightarrow D)$

Benchmarking CA-based Problem Solver on Formalized Pre-univ. Math Problems

Motivation

- Development of the AR layer of the solver in parallel with the NLU layer
- Evaluation on problems with varying difficulty
- Estimation of the computational cost of the reasoning on NLU output

Benchmark Problems: Sources

- **Ex**: 288 problems from exercise book series
  - 200 problems on geometry
  - 100 problems on integer arithmetic

- **Univ**: 245 problems from the entrance exams of seven national universities
  - Geometry, real arithmetic, pre-calculus etc. expressible in the theory of RCF

- **IMO**: 212 problems from the International Mathematics Olympiads (1959-2014)
  - All geometry and real arithmetic problems
  - Some of number theory, combinatorics etc.
  - 2/3 of the all past problems till 2014

Encoding process

- Six students (majored in math/CS) and two full-time researchers encoded the problems in a higher-order language

- Literal translation
  - Word-by-word, sentence-by-sentence
  - No inference
  - No paraphrase

Example

Let D be a point inside acute triangle ABC such that

$$\angle ADB = \angle ACB + \pi/2$$

and

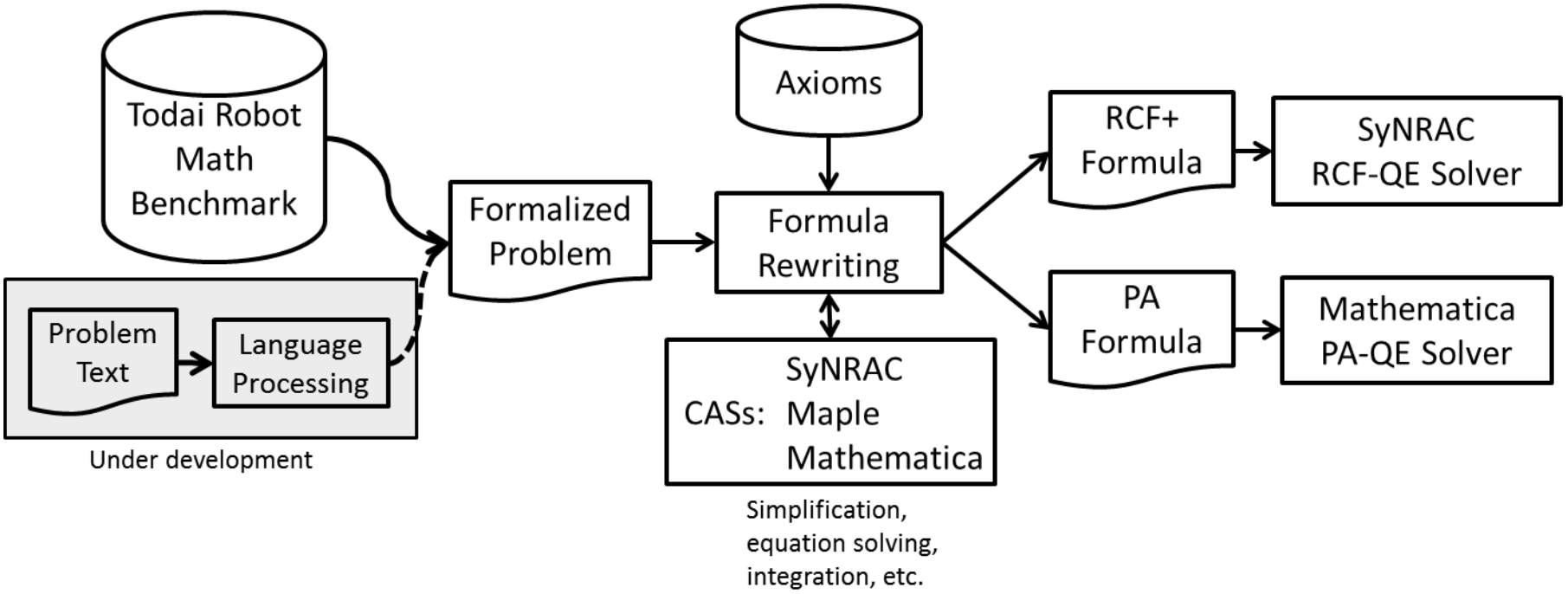
$$AC \cdot BD = AD \cdot BC$$

Calculate the ratio $(AB \cdot CD)/(AC \cdot BD)$.

(IMO 1993 Problem 2)

```
(Find (x)
  (exists (A B C D)
    (&& (is-acute-triangle A B C)
        (point-inside-of D (triangle A B C))
        (= (rad-of-angle (angle A D B))
           (+ (rad-of-angle (angle A C B)) (/ (Pi) 2)))
        (= (* (distance A C) (distance B D))
           (* (distance A D) (distance B D)))
        (= x (/ (* (distance A B) (distance C D))
                (* (distance A C) (distance B D)))))))
```

CAS-based solver



Syntactic Profile (per problem; medians)

	Pre-univ math benchmark				TPTP-THF
	Ex	Univ			
# Formulas	2	2		1	10
# Atoms	65	95	65	72	88
Avg atoms/Fml	38	54	56	48	6
# Symbols	16	19			9
# Variables	9				19
λ	3	3	1	2	2
∀	0	0	4	0	9
∃	4	6	1	4	2
# Connectives	55	78	58	61	52

Problem scale is at similar level

#of λ abstractions

Different types quantifications

Overall results

		Succeeded		Failed		
		Success %	Time (sec) Min/Med/Avg/Max	Timeout	Wrong	Other
Ex	RCF	63.8% (111/174)	13/18.0/ 37.4/ 343	10.9%	1.7%	23.6%
	PA	57.1% (48/ 84)	12/17.0/ 20.3/ 172	0.0%	0.0%	42.9%
	Other	10.0% (3/ 30)	13/14.0/ 17.7/ 26	0.0%	0.0%	90.0%
	All	56.3% (162/288)	12/17.0/ 32.0/ 343	6.6%	1.0%	36.1%
Univ	All (RCF only)	58.0% (142/245)	12/26.5/ 85.5/1417	15.5%	2.9%	23.7%
IMO	RCF	16.5% (19/115)	14/25.0/ 51.8/ 197	29.6%	0.9%	53.0%
	PA	4.8% (2/ 42)	25/29.5/ 29.5/ 34	16.7%	0.0%	78.6%
	Other	3.6% (2/ 55)	17/24.5/ 24.5/ 32	12.7%	0.0%	83.6%
	All	10.8% (23/212)	14/25.0/ 47.5/ 197	22.6%	0.5%	66.0%

- Difficulty of RCF problmes: Ex < Univ < IMO
- Difficulty of PA problems: Ex << IMO

Results on RCF problems in Ex

# of Stars	Succeeded		Failed		
	Success %	Time (sec) Min/Med/Avg/Max	Timeout	Wrong	Other
1	82.4% (28/34)	13/17.0/20.4/ 65	2.9%	0.0%	14.7%
2	79.4% (27/34)	16/18.0/28.1/230	2.9%	2.9%	14.7%
3	57.6% (19/33)	15/17.0/36.1/341	6.1%	0.0%	36.4%
4	47.4% (18/38)	15/19.0/62.1/343	23.7%	2.6%	26.3%
5	54.3% (19/35)	16/28.0/53.6/279	17.1%	2.9%	25.7%

- # of Stars = difficulty level assessed by the editors of the practice book series

Results on IMO problems by years

Years	Human Efficency	Machine Efficiency	Succeeded	Failed		
				Timeout	Wrong	Other
1959-69	58.23%	21.11%	26.3% (15/57)	22.8%	1.8%	49.1%
1970-79	46.57%	7.00%	13.3% (4/30)	26.7%	0.0%	60.0%
1980-89	44.35%	1.85%	3.1% (1/32)	31.2%	0.0%	65.6%
1990-99	38.27%	3.33%	5.7% (2/35)	11.4%	0.0%	82.9%
2000-13	34.31%	1.19%	1.9% (1/54)	22.2%	0.0%	75.9%

- Human Efficiency: IMO participants' avg. score
- Machine Efficiency: system's score

- IMO problems get harder by year both for human and machines

Summary

- Natural Language Math Solving System combining
  - Grammar-driven semantic analysis
  - Inference by QE

- Benchmark result on the inference part
  - Excercise & entrance exam: ~60%
  - Mathematical Olympiads: 5~15%